Using Novel Tasks in Teaching Mathematics: Three Tensions Affecting the Work of the Teacher

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Novel (as opposed to familiar) tasks can be contexts for students' development of new knowledge. But managing such development is a complex activity for a teacher. The actions that a teacher took in managing the development of the mathematical concept of area in the context of a task comparing cardstock triangles are examined. The observation is made that some of the teacher's actions shaped the mathematics at play in ways that seemed to counter the goals of the task. This article seeks to explain a possible rationality behind those contradictory actions. The hypothesis is presented that in managing task completion and knowledge development, a teacher has to cope with three subject-specific tensions related to direction of activity, representation of mathematical objects, and elicitation of students' conceptual actions.

KEYWORDS: argumentation, didactical contract, mathematics teaching, novel tasks, teaching tensions.

The work of teaching mathematics involves interacting with students, partnering in activities and discussions wherein mathematical ideas may emerge and be developed. The work of teaching mathematics also involves designing and foreseeing those activities and discussions on behalf of mathematical ideas that the students need to know. Interactions about mathematical tasks thus may have a double meaning for a teacher—they can be design products that respond to the intention to teach about new knowledge and also contexts in which new knowledge may emerge. As the field has tried to come to grips with the notion that students are not passive receivers but...
active builders of knowledge, that double meaning of classroom interactions has become more central. For quite some time conventional wisdom in mathematics education has indicated that by engaging students in investigating carefully chosen, novel tasks, teachers can create contexts in which students can construct meaning for key mathematical concepts that they are to acquire. Conventional wisdom has also underscored the value of students' involvement in those tasks as opportunities for participation in mathematical activity in which interesting, perhaps unintended ideas may emerge.

This article pays close attention to the way a teacher manages students' mathematical work on such tasks—tasks that respond to an instructional intention and also create a context for developing new ideas, or *novel tasks* (Doyle, 1988). The article contributes to a scholarly conversation that focuses on understanding how the various responsibilities that tie a teacher to the subject matter and to students affect the kinds of interaction about ideas that can take place between teachers and students in classrooms. Specifically, the purpose of the article is to present a theoretical argument supported by an examination of records of teaching. I argue that the need for a teacher to reconcile *responsiveness* to what emerges from students' work on a task with *accountability* to an agenda for knowledge development on behalf of which such a task is assigned creates tensions that affect the nature of the knowledge at play. I present three subject-specific tensions and argue for the hypothesis that the need to manage them regulates the work that teachers do with novel tasks. I suggest that such phenomena may be central in the work of teaching any discipline, but argue that to understand them in detail it makes sense to explore them in the context of a specific case within the teaching of a specific discipline. I explore that phenomenon of the work of teaching in the case of using novel tasks to develop knowledge of the mathematical concept of areas of triangles.

I propose that novel tasks have the potential to confront teachers with three *tensions*; the management of those tensions is a process that shapes in particular ways (and might even compromise) the ideas that are developed in the interaction with students as they work on such tasks. These three tensions concern the direction of students' activity, the representation of mathematical objects, and the elicitation of the conceptual actions that students need to invest. The three tensions are illustrated by an in-depth analysis of a lesson wherein an important new idea comes to the fore as a result of students' engagement in a novel task about the concept of areas of triangles. The three tensions, which are centrally related to the structure of academic tasks (Doyle, 1988), are used here to interpret how the teacher managed students' production of that new idea and its incorporation to the shared knowledge of the class. I explain how these tensions spring from the need of the teacher to balance the two responsibilities noted above of partner and designer of the knowledge-development enterprise that relates students, teacher, and subject matter.

The structure of the article is as follows. I start with an account of a classroom episode that I use as context to present the theme. The episode was
part of a 6-week, for-credit summer course in geometry for middle school graduates, taken by a class of four 14-year-old girls. The episode deals with a task (the “ranking triangles task”) in which the students had been working in pairs and one of them (Didi) comes up with an idea about areas of triangles. The episode shows how Didi’s teacher (Earl) manages the process of incorporating her idea into the knowledge being developed by the class. In a preliminary reading of the episode, I bring to the fore those actions of Earl that contributed to shaping and—from a mathematically oriented observer’s perspective—distorting Didi’s idea. As I provide that reading, I outline three critical issues in Earl’s management of Didi’s idea. I suggest that these critical issues can be explained by hypothesizing that the teacher is managing the three subject-specific tensions alluded to above—thus uncovering a plausible rationality behind the actions of the teacher. I subsequently outline a theoretical framework built around the notions of academic tasks (Doyle, 1983, 1986, 1988) and the didactical contract (Brousseau, 1984, 1990, 1997), and around the characterization of the teacher as dilemma manager (Chazan & Ball, 1999; Lampert, 1985). I use the theoretical framework to achieve a detailed description of how the interactions between Earl and his students allowed the emergence of Didi’s idea and shaped its meaning. Then I use the three identified tensions to explain the role that Earl played in shaping the meaning of Didi’s idea. Finally, I come back to proposing that the three tensions that help in understanding Earl’s actions are indeed useful in understanding the use of novel tasks in the work of teaching.

Episode: A New Way of Using the Area Formula

Earl started the first day of the instructional unit on area of plane figures with a brief discussion about areas of triangles—leading his students to recall the formula² that they had learned in middle school. He then assigned his students to work in pairs and gave each pair a set of eight triangles cut out in cardstock (see Figure 1) and a toolbox.³ He proposed the following task:

I'm gonna give you guys, each pair, a set of eight triangles and your job is gonna be to compare and rank them. Ultimately what we want to do is be able to order them or rank them. But there is a catch. What we'll try to do is have you do the ranking without using the area formula or with using it as little as possible. So your challenge is to minimize the use of the formula.

Each group had also been given a chart like the one offered in Figure 2 where they had to compare pairs of triangles and provide reasons for each comparison. As students worked on this task, Earl moved around the room, talking to each pair of students. By the time Didi made her conjecture, both pairs had independently discovered various properties of area as they compared the triangles. They had used the idea that if a triangle “fits inside” another one, then the area of the former should be less than the area of the latter. They had also realized that if a triangle was cut into pieces and the
Figure 1: Shapes provided for the ranking triangles task.

pieces rearranged without overlaps, the covered area would not change, and this could be used in connection with the “fit inside” idea.

As Didi and Gina took on comparing triangles D and E (see Figure 3), none of these previous strategies proved helpful. But as they compared bases and heights, they discovered that one side of E doubled one side of D and

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<th>Comparison</th>
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<td>Triangle ____ is larger than Triangle ____</td>
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Figure 2. Comparisons and reasons sheet provided for the ranking triangles task.
Figure 3. Triangle D (of vertices M, L, and K) has base MK and height LH. Triangle E (of vertices M, O, and N) has base MN and height OP. Base MK is twice as long as base MN, and height OP is twice as long as height LH.

That the corresponding height of D doubled the corresponding height of E. When Earl approached their table, Didi shared with him their observations:

Didi: The base of this [points at triangle E] is double the base of that [points at triangle D] and then like pretty much like the altitude of this [points at triangle D] is double the altitude of that [points at triangle E].

Gina: So she thinks that makes it equal but I don't . . . because this base [points at triangle E] is greater and this height [points at triangle D] is greater, how can they be equal?

Earl asked Didi to share her thinking.

Didi: Like equals . . . it cancels . . .

Earl: [Smiles] Are you relating that to a formula, just to a feeling, or . . .

Didi: [Chuckles] I don't know, I guess to a feeling . . . well if you do . . . base times height . . .

Gina: You are not allowed to use the formula.

Earl suggested that they could use strategies they had been using before but to no avail. He indicated that “this would probably be a reasonable time to try the formula and see if it confirms Didi’s conjecture.” He asked them to “write it out, that they are equal. And basically your conjecture is that if one has double the base and the other has double the height then they should be equal.”

When Earl came back to the group, Didi and Gina had measured bases and heights and calculated their areas using the formula. They reported that
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they were not equal, as triangle $D$ was 46.1 (cm$^2$) but triangle $E$ was 44.6 (cm$^2$). Earl inspected the original measures and made the point that, whereas the claim that the areas were equal had turned out to be false, it was also the case that the grounds for the argument had not been true either. Earl suggested that “maybe [they] should pretend they [were] double and calculate to see if [Didi’s] hypothesis [was] right . . . but if they [were] not double then that [would be] a different story.” Earl insisted that “to test Didi’s hypothesis you need to assume that measurements are really double.” He asked them to “try [the height of $E$] as 9 and [the base of $E$] is 10 and a half, and [the base of $D$] is 21 and [the height of $D$] is 4 and a half, and do the calculation to see if Didi’s [conjecture is valid].” After they did so, he underscored the fact that the calculations with those numbers yielded equal areas: “So that suggests that Didi’s conjecture could be right . . . we haven’t proved it of course. . . .” But he came back to qualifying the import of the conjecture for the situation at hand: “Now that doesn’t mean these two triangles are the same, especially with these cutouts, it’s not exact.” He then recommended that they try a completely different idea for comparing those triangles.

An Initial Reading of the Mathematics at Play in This Episode

The mathematics that is involved in students’ actions and discourse in this brief episode is extremely interesting. The task of ranking triangles according to area provided a context for a student to make a claim about what the relationship between two areas should be, basing that claim on mathematical rather than empirical reasoning (Reid, 2002). As they compared triangles $D$ and $E$, in particular, the empirical strategies students had previously used (such as cutting up a triangle and placing the pieces inside another) were deemed insufficient. In response to the need to compare those areas without calculating areas, Didi’s idea suggested that their knowledge of the ratios between the bases and heights of the triangles meant that they could know something about the relationship between the areas—in this case, that these areas were equal. Didi’s claim was counterintuitive (the triangles looked so different that it was hard to imagine them being equal), and yet there was reason to anticipate that they might be equal. Didi gave reason for her conjecture through a virtual use of the area formula—the factors would cancel if one were to use the area formula. This was an implicit yet novel way of using the area formula, not as a device to obtain an area but as an algebraic expression that could stand for (be used in lieu of) that area. That seems to be the starting point—a “fragile” starting point indeed (Brousseau & Otte, 1991)—of a fine piece of mathematics. One could foresee the development of a discussion about how ratios between the bases and heights of triangles determine ratios between the areas of those triangles, even when the areas themselves remain unknown.

An observer of this episode might note, however, that such development of mathematical propositions about area did not go beyond that fragile beginning because of some critical problems in the way Earl managed the
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work. The observer might lament that Earl so quickly directed students to calculate the actual areas that, as a result, they discovered too early that the conjecture was not relevant to the particular comparison they were making. The observer might also lament the fact that Earl and his students ended up merely exemplifying Didi's conjecture with ad hoc numbers, instead of representing the problem algebraically, in which case they might have been able to prove the conjecture. The observer might further lament that the discussion about Didi's conjecture was not pursued: Instead of working on other possible properties of area that used the area formula in the same novel way, the class continued to compare the cardstock triangles given at the outset. In summary, an observer might suggest that Earl's actions shaped as very limited in validity and scope an otherwise important idea emergent from the work that these students were doing. Didi's conjecture, in particular, remained a numerically verified, irrelevant claim whose reasons were still hidden and whose generality remained unexplored.

Whereas it may be true that Earl could have done things differently, I want to suggest that Earl's actions are understandable—though not necessarily optimal or unavoidable. It is possible to appreciate in Earl's actions the difficulty that a teacher would have in supporting the production of, and dedicating attention to, Didi's novel way of using the area formula and the conjecture she produced. That difficulty deserves explanation, because it brings to the fore tensions that relate to the nature of any novel task and its relationships with the new ideas that such a task might elicit. In the following section, I provide some elements of the theoretical framework with which I look at this episode, after which I state the three tensions that this episode reveals.

Theoretical Framework

The theoretical framework captures the difficult position in which a teacher works; having to respond to an institutional agenda for knowledge development that leads him or her to create contexts for learning and to the intellectual bids that students produce as they work in those contexts. The three elements of theory that I outline subsequently provide the basis upon which I build my own theoretical contribution to understanding the work that teachers do as they undertake the process of using novel tasks to teach.

The Didactical Contract and the Teacher-Student Relationship

The mathematical ideas that students come to know in classrooms are shaped in and through the interactions in which they participate (Bauersfeld, 1988; Cobb & Bauersfeld, 1995; Voigt, 1985, 1994). As teacher and students interact about a task, they create meaning for the mathematical ideas at play in that task. In that approach, the teacher is a partner with the student in the local production of mathematical meanings. However, as one considers the institutional position of the mathematics teacher, a complementary description
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is in order that relates the teacher with an institutional project of knowledge development. In that description, the teacher is responsible for the design of experiences in which students can learn about specific mathematical ideas that are part of an agenda. The theory of the didactical contract provides a way of understanding the difficult position of a teacher who, at the same time, shapes mathematical meanings as he interacts with students about their mathematical work and designs those interactions with the intention that students come to know about specific ideas.

The expression "didactical contract," coined by Guy Brousseau (1984, 1997), suggests the existence of implicit norms that operate like a contract, regulating the three-way interactions among teacher, student, and the subject of studies, mathematics in this case (Chevallard, 1985; Cohen & Ball, 1999). The large print of this contract is set from the moment that teacher and student come together to deal with the subject matter. It establishes a mutual obligation between teacher and student with respect to mathematics: The teacher must teach mathematics to the students, and the students must learn mathematics with the help of the teacher (Brousseau & Otte, 1991). Brousseau postulates that the specific clauses of such a contract—the fine print, so to speak—are permanently being negotiated at the same time that they are enforced. Such negotiation of the didactical contract allows teacher and student to work together and at the same time fulfill their respective roles as they do their share of work.

The phenomenon examined in this article is an example of a particular kind of negotiation of the didactical contract: the negotiation of responsibilities between teacher and students regarding the production and recognition of new ideas in the context of novel tasks. The notion that as participants interact over a task, they negotiate the fine print of the didactical contract is used to describe how they shape the ideas at play. The notion that teacher, students, and subject matter are bound by a didactical contract that they must not breach is the source of explanations for the negotiation described. In particular, I describe the negotiations that led to the production and recognition of Didi's conjecture. And I show that the hypothesis that three subject-specific tensions operate on the teacher's management of a novel task is a way to explain those negotiations and the way in which the ideas at play were shaped.

Mathematical Tasks and Mathematical Knowledge

Given the present focus on novel tasks, it is pertinent to comment on the relationship between task and knowledge with regard to students' learning of mathematics in classrooms. As Doyle (1988) indicates, the tasks that students work on in classrooms constitute key contexts for students' thinking about the subject of their studies, in this case mathematics. According to Doyle (1988, p. 169), academic tasks have four general components: (a) a product to be obtained, (b) operations to produce that product, (c) resources to be used in that production, and (d) the significance of the task in the accountability system of a class. To investigate the teacher's management of
novel mathematical tasks, I take these components as indicators of four things that need attention as the negotiation of the didactical contract is examined: (a) the product announced to students as well as the products expected or recognized by the teacher; (b) the conceptual actions directly indicated to students as well as those that students might develop themselves to undertake the task; (c) the actual representations or embodiments given as resources as well as the possible ways in which they could be used or transformed; and (d) the way in which students' work on the task relates to their customary obligations to the teacher and to the subject of studies.

These indicators are especially helpful when one thinks about the use of novel tasks (Doyle, 1988) as contexts for students to develop new ideas. Novel tasks may identify certain products, resources, and operations but aim at others, unspeakable at the moment at which the task is issued. In particular, novel tasks may aim at students coming up with and using new ideas, ideas on behalf of whose development the task had been chosen in the first place (Brousseau, 1997, chap. 1; Douady, 1991). Managing the use of these novel tasks to fulfill well-defined instructional purposes is an enterprise intrinsically difficult for a teacher (Henningsen & Stein, 1997; Stein, Grover, & Henningsen, 1996).

Tensions and Dilemmas in the Work of Teaching

To think about the teacher's difficult work of managing task enactment and knowledge development, I draw on an important domain of scholarship on teaching that has put forth the notion that the work of a teacher is affected by implicit tensions (Cohen, 1990) as well as conscious dilemmas (Lampert, 1985). Rather than being simply personal deficiencies of individual practitioners, these dilemmas and tensions are endemic to the work of teaching (Ball, 1993). Magdalene Lampert (1985), for example, explained how a tension between the need to maintain order and the commitment to generating gender-equitable opportunities to learn created a dilemma for her as a teacher regarding how to coordinate the management of physical space and classroom discourse as she taught. Lampert argues that such tensions cannot be handled as one-time decisions. Rather, they have to be managed constantly as a teacher teaches; they are dilemmas. Daniel Chazan and Deborah Ball (1999) have also shown that those tensions and dilemmas that affect the work of teaching mix elements that are generic with elements that are discipline specific (see also Chazan, 2000; Lampert, 2001).

As noted in the rendition of the episode under consideration, the play of mathematical ideas in the context of a task appears to be very sensitive to the actions of the teacher. The notion that while acting in a certain way, a teacher may be coping with tensions is crucial to making sense of why those actions take place. My purpose is to use the hypothesis that Earl was managing some specific tensions in order to craft a reasonable explanation for why he took the actions that he took in shaping the mathematical ideas at play in the episode. I do not, however, make claims about what went on in
Earl's consciousness as he was teaching—my claims amount to saying that "everything happens as if" (Bourdieu, 1980) Earl's actions resulted from his need to manage this or that tension. The validity of tensions is predicated on their usefulness to interpret action in context. The identification of possible tensions in a complex episode can help us understand the relationships between tasks and new knowledge and foresee the complexities involved in a teacher's use of novel tasks to teach.

Main Thesis: Three Tensions to Manage
While Using Novel Tasks to Teach

The bulk of this article consists of an examination of a case in which a teacher and his students negotiate the conditions to produce a new item of knowledge (Didi's conjecture) in the context of their work on a novel task (the "ranking triangles task"). The production of Didi's conjecture hinges on an emerging idea whose legitimacy is contested—that the area formula might be used to predict without calculating. The case is examined to demonstrate the relevance of three hypothetical tensions in interpreting the actions of a teacher managing the treatment of new ideas that emerge from students' work on a task. The tensions that I propose are thus the following three, which I here tie to the main characteristics of an academic task provided by Doyle (1988).

The first tension refers to where to direct students' activity. The assignment of an academic task to students includes identifying for them a product that they are to obtain. However, whereas a novel task explicitly targets one such product, the most important "product" of the task may be hidden from the students' view when they take on the task. As a teacher manages students' engagement with a novel task, two opposing obligations may create a tension for him or her. On the one hand, the teacher may be compelled to maintain students' attention to the product that was explicitly set as an expectation of their work. On the other hand, the teacher may be compelled to seize the possibilities for the development of new ideas that students' work offers to him or her. I contend that these two obligations can be present at the same time and create a tension for the teacher to manage. This tension is illustrated in the episode under consideration by Earl's need to pay attention at the same time to the outcomes of the comparison of triangles and to Didi's conjecture and argument.

The second tension refers to how to represent mathematical objects. The assignment of an academic task involves making resources available for students to use in obtaining the product. These resources include representations or embodiments of mathematical objects. The nature of these representations is crucial for students' work if the context is a novel task in which they are to produce new understandings. As the teacher envisions and manages students' work with those representations, two opposing obligations may create a tension for him or her. On the one hand, a teacher may be compelled to identify precisely which features of the representations used in the
task are relevant to the mathematical ideas targeted by the task. On the other hand, the teacher may be compelled to maintain a certain degree of vagueness regarding what in those representations is relevant so that students are reserved the opportunity to mathematize, to make deliberate choices instrumental to their inquiry. I contend that these two obligations can be present at the same time and may create a tension that the teacher needs to manage. This tension is illustrated, in the episode under consideration, in Earl's struggle with directing students' attention to the mathematically interesting features of triangles for which Didi's conjecture would be true and the real features of the cardstock triangles that were actually implicated in the comparison task.

The third tension refers to how to elicit students' conceptual actions that are instrumental for the task. An academic task involves operations that students undertake in order to arrive at the product. These include physical and conceptual actions that are instrumental in completing a task. In particular, part of what students might gain from their work is the opportunity to shape new operations that become meaningful in the context of a novel task. As the teacher manages students' work on such a task, two opposing obligations regarding those operations may affect his or her work so as to create a third tension. On the one hand, the teacher may be compelled to give students unambiguous directions and constraints that indicate to them what they are expected to do and think about as they work on the task. On the other hand, the teacher may be compelled to maintain a productive ambiguity about directions and constraints, keeping the task open for students to come up with actions that make sense to them as being instrumental in completing the task. I contend that these two obligations can be present at the same time and may create a tension for the teacher. This tension is illustrated, in the episode under consideration, in Earl's way of coping with Gina's objections to Didi's use of the area formula—which had been discouraged at the moment of launching the task.

In the next section, I show how I inspected the task and the episode under consideration in order to describe how Earl and his students negotiated the didactical contract while working on the ranking triangles task. Then I report on that inspection, tracking those of Earl's actions that were crucial in enabling and shaping the emergence of Didi's conjecture and argument in the context of the ranking triangles task. In the last section, I show how the three tensions outlined above help us understand Earl's actions in that negotiation process. In addition, I provide a general discussion of these tensions. I propose that whereas these tensions are visible in not-so-ordinary situations (such as the analyzed episode), they have more widespread, predictable effects in how practitioners accommodate the recommendation of using tasks to teach (National Council of Teachers of Mathematics, 1991). The tensions are ways of organizing an important chunk of the multiple obligations that are at stake for a teacher when he or she tries to involve students in the development of new ideas.
Method

This article aims at using "records of practice" (Lampert & Ball, 1998) as experiential grounds to explore an intellectual problem and discuss some theoretical ideas that might shed light on that problem. In the following sections, I describe how these records of practice were collected and inspected in order to create those experiential grounds.

Data Collection

The records of practice used to examine my questions about the work of teaching were collected as part of a weeklong classroom teaching experiment (Cobb, 2000) that I conducted in collaboration with Earl (a pseudonym), an experienced high school and middle school mathematics teacher. The general focus of the teaching experiment was to examine what it would take for a teacher to get students to make and prove conjectures in a domain of geometry where the truth or falsity of those conjectures was not evident from looking at diagrams. I had been investigating this question in various other collaborations with teachers, targeting conjectures and proofs in the domain of areas of plane figures.

Earl had accepted a job teaching a 6-week-long summer course in geometry for students who had just finished middle school; the course would provide a one-semester credit in high school geometry. Earl had chosen Michael Serra's (1997) Discovering Geometry textbook for this class because its purported inductive approach consisted of engaging students in hands-on activities as contexts to develop the ideas of geometry. For the unit on area of plane figures, Earl and I had agreed that he would depart from the text and instead use some alternative tasks that I would prepare. As Earl looked at samples of these tasks, he indicated they were similar to those in the text and that he would feel comfortable using them in his class. The goal of the tasks was to have students make and prove conjectures about the areas of figures using the geometric properties of those figures, and in doing so to come to know new things about area (Freudenthal, 1983; Lakatos, 1976). As he agreed to work with me, Earl understood that my interest was not in evaluating the implementation of a curriculum but rather in examining the work of teaching involved in using certain kinds of tasks as contexts for students' development of new ideas. Thus, whereas Earl would commit to use the tasks that I created, he was also encouraged to adapt them as much as needed to make them viable.

Records gathered included an initial conversation about the general approach for the unit and a discussion of an initial sequence of tasks that I had created. These conversations delved into the mathematical significance of the tasks and their intent as instruments to bring about ideas of area. I was interested in using these tasks as contexts to inquire into the work of a teacher, and for that purpose it was important to maintain a sense that tasks are for a teacher both instrumental and open-ended; thus, as we discussed the ranking triangles task, I was careful in not specifying for Earl how each
particular comparison might yield to developing a particular idea of area. We rather talked about all possible ideas of area that might emerge from the joint work of teacher and students in the ranking triangles task as a whole. I interviewed Earl before each lesson to discuss aspects of planning that mainly included his anticipations of how the work would unfold. I also interviewed Earl after each lesson to discuss those anticipations as well as debrief on how the lesson had gone. Our conversations before and after each class, used for planning and debriefing, were audiotaped, and Earl also kept an audiotaped journal as the week went on. The lessons themselves were recorded with two videocameras, one of which followed Earl as he moved across the room and zoomed into his interactions with students or groups; the other camera was stationary, capturing whole-class scenes.

The records used to inspect the issues addressed in this article pertain to the first lesson of this weeklong experiment when students worked on the ranking triangles task. The main source of data for this article was footage captured by the camera that followed Earl as he interacted with his students. Footage was parsed into segments according to a combination of the “scene coordinates” that Doyle (1986, p. 397) uses for “identifying segments.” Because the focus here is on Earl’s actions, boundaries between segments follow closely changes in scene for him. Earl moved frequently from pair to pair, and the footage inspected focused on him; thus, each time he moved on to speak with a different pair of students represented the beginning of a new “segment.” Parsed in that way, the 90-minute lesson examined contained 32 segments, which are summarized in the Appendix.

Earl’s interactions with Didi and Gina, as well as his addresses to the whole class, are the nucleus of the empirical corpus for the examination of what Earl did with Didi’s conjecture. Transcripts of interviews before and after the lesson and of Earl’s journal provided supporting information that was used to confirm that the issues included in the formulation of the tensions were indeed relevant to Earl. Interviews did not seek to get confirmation from Earl that he was consciously managing those tensions; there were fundamentally two reasons for not doing that. The first reason is conceptual: Whereas dilemmas and tensions are at times conscious (Lampert, 1985), it is also possible that the simultaneity of responsibilities (Doyle, 1986) to which a teacher must attend makes it difficult to maintain clear records of the reasoning processes he or she is conscious about while acting. From this perspective, confronting Earl with the propositions that he had indeed been managing these tensions could as much imply an invitation to verify my claims as it could imply administering a probe to elicit yet a different kind of data. In the absence of theoretical means to interpret his possible responses to such confrontation, and on account of my interest in crafting these tensions not as descriptions of a true rationale but as explanations of a plausible rationality, I chose not to confront Earl with the tensions crafted. The second reason is of an ethical nature. For a practitioner to have to contend with (recognizing or denying) this choice of language and logic in accounting for his own practice might have professional or emotional implications for
whose goodness or usefulness I knew no warrant, even though as a scholar I find it compelling to explain instructional actions as tension managing.

Data Analysis

To gather the evidence necessary to illustrate the points made in this article, the data were examined in three stages. The first stage consisted of an a priori analysis of the task, examining the task prior to its enactment in order to identify the ideas that might arise as students undertook the task and the particular difficulties that the task might present to students. The second stage consisted of examining actual records of the interaction between teacher and students during their work on the task in order to trace how the didactical contract was negotiated to develop and treat some of the new ideas that emerged. In particular, this examination focused on tracing those of Earl’s actions that mattered in that negotiation. The third stage consisted of examining records of classroom interaction and interview transcripts in search of pieces of evidence that connected Earl’s actions with the various obligations that I later used to construct each of the explanatory tensions I present.

Analysis of the task planned

The first step of the analysis consisted of examining the ranking triangles task a priori (i.e., in terms of what it might have elicited; see Comiti, Grenier, & Margolinas, 1995; Herbst, 2002; Stein et al., 1996). This analysis allowed me to identify some “observables” with which to examine the records of the enactment of the task. Data used in this analysis a priori included the plans made for the whole unit and the particular lesson where the ranking triangles task was used and the actual materials prepared for the task. I used Doyle’s (1988) components of a task to inspect each of the pairwise comparisons of triangles that students would need to make in order to complete the ranking triangles task. This analysis consisted of anticipating possible operations and resources that students might use to make the various comparisons. It served as a background to appraise what it would take for a teacher to support the production and public treatment of Didi’s conjecture. The result of the analysis was a set of observational questions that I used in the second stage, to inspect the video records related to Didi's conjecture. These questions would help me identify moments through the lesson where pieces of Didi’s conjecture emerged (beyond the punctual episode described above) that might provide evidence of how the didactical contract had been negotiated so as to enable the conjecture to exist and be treated.

Analysis of the negotiation of the didactical contract

The second phase consisted of using the anticipations made in the analysis of the task to inspect the records of the enactment of the task (Stein et al., 1996). This analysis of the enacted task aimed at answering the question of
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how Earl and the students negotiated their mutual responsibilities in the
didactical contract so as to enable Didi's conjecture to emerge and receive
treatment in the context of this task.

I closely inspected records captured with the camera that followed Earl
as he interacted with each group of students. As Didi's conjecture emerged
in the context of Didi's and Gina's collaborative work comparing triangles \(D\)
and \(E\), my analysis involved inspecting very closely the interactions between
Earl and these students. I looked at the segments when Earl and these stu-
dents discussed that particular comparison (segments 10, 12, 15, and 17; see
the Appendix) as well as at other segments (notably, segments 5, 6, and 8)
where preliminary elements for the comparison of triangles \(D\) and \(E\) were
discussed. I also looked closely at segments 3 and 4, in which Earl presented
and explained the task to the whole class, and segments 30 and 31, in which
Earl discussed with the whole class some comparisons and Didi's conjecture
in particular (see the Appendix for a succinct description of what those and
other numbered segments contain). I inspected the interactions between
teacher and students in these segments using analytic induction (LeCompte
& Preissle, 1993).

I coded segments of conversation according to how each segment con-
tributed to shaping the negotiations regarding the product, the resources,
and the operations that constituted the task. The elaboration on a task's com-
ponents outlined in the theoretical framework was used as an initial set of
codes to note the orientation that each segment was providing to the various
issues of interest and in particular how Earl was contributing to such orien-
tation. Thus, in terms of product, discourse within each segment was initially
classified as either maintaining attention to the comparison of triangles or as
shifting attention to the more abstract properties of the area concept at play
in making specific comparisons. In terms of resources, discourse was initially
classified as either maintaining attention on the empirical characteristics of
the real objects provided at the outset (the cardstock triangles) or develop-
ning more abstract representations of the mathematical objects at play (the
quantitative dimensions of a triangle that matter for area comparisons). In
terms of operations, segments were initially classified either as maintaining
the task open for students to use comparison strategies that made sense to
them or as further specifying for students what they should, could, or should
not do to complete each comparison.

The exchanges extracted from the interactions between the participants
were organized as elements of two story lines that I hypothesized as devel-
oping simultaneously as the task was being investigated. These story lines
were a baseline story (about what was being done to complete the assigned
task) and an emergent story (about what novel ideas on the concept of areas
of triangles were coming to the fore as students worked). As I extracted those
utterances and classified them in terms of referring to the completion of a
task or to the generation of ideas, I sought words that indicated interaction
(conflict or cooperation) between the two story lines.
Analysis of tensions managed by the teacher

I used those places in the discourse (from classroom interaction as well as from conversations with Earl) where I could detect interaction between story lines to identify obligations that could be traced back to what the didactical contract imposes on the teacher and that appeared activated in the situation of using a novel task to teach. As the teacher managed the development of the two story lines (completing the task and developing knowledge), he would make comments that pointed to issues more general than those story lines, such as maintaining student engagement and assigning hands-on activities. He would also refer to issues such as those we planned for or debriefed about the lesson. The notion that the didactical contract involves the teacher in responsibilities toward students as well as toward mathematics, and that those responsibilities shape and are shaped by the various components of a task, served as an initial heuristic to find those references. I then aggregated references to these general issues according to how each of them seemed to influence the way the teacher participated in the negotiations of the task—particularly in negotiating the product, the resources, and the operations involved in the task. Thus, the various responsibilities or obligations that I later used to craft explanatory tensions were all confirmed empirically (through evidence found in Earl's discourse with his students or in the interviews). I organized these responsibilities into statements of tensions so as to explain why the teacher would participate in the negotiations of each component of the task in the way he did. These tensions are elements of a theoretical model that is validated insofar as it is useful to uncover a plausible rationality for Earl's actions. The three tensions that I propose in this article are the outcome of that process.

The Task and the Negotiation of the Didactical Contract: An Inspection

The Chosen Task as a Context for Knowledge Development

In this section, I provide an account of the design of the ranking triangles task. I situate the specific comparison of triangles D and E in that context. I argue that there were specific characteristics about that comparison that made it an opportunity for the emergence of a new way of using the area formula. Yet, because developing that new idea presumed that students would do some things very differently from what the task explicitly asked for, I anticipate that the treatment of that new idea might require special negotiations of the didactical contract.

The ranking triangles task and the concept of area of triangles

The envisioned statement of the ranking triangles task was quoted at the beginning of the article—Earl introduced it exactly as planned. The task identifies a final product (to come up with a ranking of the eight triangles accord-
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ing to their area) and a set of intermediate products (to compare triangles pairwise according to their area, providing justifications for each and reporting them on a worksheet; see Figure 2). Resources for the task included the cardstock triangles as embodiments of the mathematical notion of triangle as well as the various tools in the toolbox.

As far as the operations that students had to carry out to obtain the product, it is helpful to examine those in contrast with what students would have likely done had they not been issued the challenge of not using the area formula (hereafter the constraint). Students knew the formula \( A = \frac{1}{2} \cdot b \cdot h \) as a device that yields the actual area of a triangle when actual numbers are substituted for \( b \) and \( h \). They had used that formula in middle school in tasks that, given a triangle, require students to obtain its area. If the need to use the area formula had not been challenged, the ranking triangles task would have entailed a straightforward set of familiar operations: measure bases and heights for each triangle, calculate their areas, and rank these numbers. The fact that this version of the task would have been familiar to them was indeed confirmed later in the interactions. The given task was, however, novel because Earl imposed the constraint on the use of the formula and because he did not indicate what operations students would have to follow to achieve their product. Rather, students were expected to grab several properties of triangles and area and use them to effect and justify comparisons. Some of those properties would likely be new only insofar as being explicitly formulated. For example, the notions of (a) inclusion, \(^{10}\) (b) break-and-make (see Footnote 4), and (c) transitivity\(^{11}\) appear very early as operational invariants in children's construction of area (Piaget, Inhelder, & Szeminska, 1960), but the requirement that students justify their comparisons might result in the statement of those operational invariants as explicit properties of area. The ranking triangles task also provided a context for other possible properties to emerge as tools to decide on a comparison. Particularly interesting tools that might emerge were the implications of the multiplicative nature of the triangle area formula, for example:

- If two triangles have the same base and the vertex opposite the base in each is on a parallel to the base, their areas are equal.
- If two triangles have congruent bases (or heights), their areas differ by the ratio in which their heights (respectively, bases) differ.
- The areas of two triangles stand in a ratio equal to the product of the ratios between their bases and the ratios between their heights.

Thus, the challenge to minimize students' use of the area formula (which for them tacitly meant minimizing calculations of actual areas) pointed to this task as a novel one. But the stated product of this task did not explicitly suggest that they might be discovering something new—if anything, it might suggest that they would have to make an effort to draw on other things that they already knew about area. The ensuing analysis of the operations involved shows why the task was indeed a novel one for these students.
Comparison of triangles D and E and the "expressive" use of the area formula

The comparison of triangles D and E, perhaps more than other subtasks, created a context for students to realize that the triangle area formula not only yields a number, but also expresses a relationship between linear dimensions of a triangle (base, height) and "the amount of stuff inside the triangle." Thus, \( \frac{1}{2}bh \) stands for the area of a triangle whose base is \( b \) and whose height is \( h \), even if for some reason one may not choose (or be able) to multiply values for those quantities and obtain a number. Triangles D and E were special insofar as students' chances to decide how those triangles compared would be dramatically enhanced if they realized that the area formula could be used in that novel way.

The concrete triangles D and E were such that thinking about quantitative relationships between bases and heights and considering how they would mingle in the expression of the area formula might provide more purchase than using other strategies (such as break-and-make). Thus, whereas the product and resources of this subtask were similar to those of others in the ranking triangles task, the operations that one might have to do to arrive at a product (a comparison that could be justified) were different. Strategies that would work for other comparisons might take longer to implement and remain inconclusive in this case. Faced with the need to compare D and E, students would not likely have perceptual means to anticipate an outcome. Had they not been issued the constraint, this comparison would have been a clear case in which measuring bases and heights, calculating areas, and comparing numbers would seem the only option available to them.

The constraint would thus discourage the calculation of area without necessarily casting doubt on the perception that the formula was needed. As triangles could be manipulated, students might juxtapose and compare bases and heights in order to feed a virtual use of the area formula. Students might accommodate some of the dimensional comparisons into expressions of the area in anticipation of what the formula should yield. Thus, students might handle D and E by first finding reason to support the claim that the base of D and the base of E were related by some ratio. They could make a similar claim about the heights of those triangles. They could then plug those ratios into the formula and anticipate what the formula might yield. The numerical values for those areas might remain unknown, but on the basis of the relationships between the dimensions of the figures one might anticipate that they should be equal. An important conclusion of examining the possibilities afforded by the comparison of triangles D and E is thus that the following new idea might be developed. Students might come to the realization that the area formula can be used as an instrument to express (not just to calculate) an area, and hence to combine known relationships between unknown dimensions of triangles and anticipate likely relationships between the areas of triangles.

Whereas the constraint afforded students an opportunity to create a new way of using the area formula, it might also function to preclude that cre-
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ation. The fact that the teacher would impose the constraint might convey to students the expectation that all comparisons could and should be done without using the area formula at all. Therefore, to produce the new idea students would have to break with the original norms within which the task had been issued. As a consequence, this a priori analysis is crucial to bring to the fore the problems that the teacher might have to address as he juggled with the task and the ideas on whose behalf the task was chosen. Those problems are in the realm of the negotiation of the didactical contract.

Anticipated issues in the negotiation of the didactical contract

The differences noted between the subtask of comparing triangles $D$ and $E$ and the rest of the task permit us to foresee that there would be a need to negotiate the didactical contract. The analysis of the task has shown how the task might elicit from students some mathematical ideas about area that were among those that Earl had to teach. Students could get to some of these ideas through observing the constraint that Earl had imposed at the outset. But be able to use the comparison between $D$ and $E$ to come to the realization that the formula for the area expresses a relationship between quantities that can be used to anticipate outcomes, a student would have to breach the initial agreement of observing the constraint. This breach might arise in two ways.

First, the student would have to choose a special representation for the objects involved—describing triangles in terms of bases and heights and the quantitative language of “double the base” and “half the height”—that was not among the resources provided. In order to choose that system of representation, a student would have to choose to treat geometric objects (e.g., the side of a triangle taken as a base) as quantities at the same time that she or he chose not to treat them as numbers (e.g., the actual length of such a segment).

Second, the student would have to decide that using the area formula to express a relationship is a different kind of use than what was alluded to by the constraint. The student would have to choose to use it to solve the problem in spite of the possibly unsafe nature of that decision. In other cases (when two triangles have equal bases and different heights), the student might have operated only implicitly with the formula as an expression of a relationship, thus not being aware that she or he was actually using the area formula. For the comparison of $D$ and $E$, however, the student would have to do some explicit calculations, such as simplifying coefficients (see Footnote 12). The enticement to do so on grounds of intellectual relevance might openly conflict with the perceived expectation that the student should abide by the constraint.

The point in noting these two possible breaches in the didactical contract is to say that whereas comparing triangles $D$ and $E$ affords the chance that somebody might see the area formula in a novel way, there are two crucial obstacles in the way of coming to such a realization. Opportunity and obstacle are both in place as flip sides of the constraint. The task is a context where students might find it useful to represent and solve a problem.
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using algebra but does not indicate that they should do so; instead, the task discourages the use of the area formula. As a result, even if the students would find it interesting and compelling to solve the problem by using the area formula in a new way, they might also see that action as out of place, as a transgression. To argue for it, students would have to articulate their choices against the indications of the task, would have to persuade their partners, and would risk being wrong or missing the target of the task. In sum, the students might be liable of breaching the didactical contract. Thus, for a student to be able to make public his or her thinking along the lines of the opportunity afforded by the comparison of triangles \( D \) and \( E \), and for the teacher to take the opportunity to develop those ideas, the didactical contract would have to be negotiated.

Hence, the a priori analysis of the comparison of \( D \) and \( E \) provided two pointers for the inspection of records of the enactment of the task insofar as these related to Didi's conjecture. On the one hand, there was a need to trace how the didactical contract had been negotiated so that students would use the language of the area formula to talk about the triangles. On the other hand, there was a need to trace how the didactical contract had been negotiated so that students would decide on relationships between areas through comparisons of the linear dimensions of triangles. In the next section, I report how the interactions surveyed with the assistance of these two pointers provided a perspective on what it took to produce Didi's conjecture in the context of comparing triangles \( D \) and \( E \).

Tracing the Negotiation of the Didactical Contract

In this section I outline how Earl and two of his students negotiated a didactical contract that permitted the production of Didi's conjecture in the context of the ranking triangles task. The description includes showing how something distinct (something that would be worthy of being called a conjecture) emerged in the context of making a claim about how the two triangles compared, and how Earl and his students dealt with this conjecture.

I organize the discussion of the negotiation of the didactical contract into three moments of the baseline story of comparing triangles. The emergence and treatment of Didi's conjecture is represented as the intrusion of a second, emergent story into the order of events of the former. The first moment deals with preliminary work that the two students did with the two triangles. The second moment deals with Didi's claim \(^{14} \) that both triangles have the same area. The third moment deals with how they contest and falsify Didi's claim. In each of these three moments, I focus on the elements of the emergent story that come to the fore in order to shape Didi's conjecture.

First moment: The groundwork for the comparison and for Didi's conjecture

For Earl, a main mathematical goal of the lesson was for students to realize that the concept of area exceeds the use of a formula to find what areas are. As he told me before class, that would provide them a "deeper under-
standing of ... area of triangles as opposed to just half times base times height. If students could formulate and use intuitive strategies such as inclusion and break-and-make, that would mean for him that they had developed some of that deeper understanding. In his initial interactions with the groups, he tried to get students to verbalize those strategies, picking up on the words (e.g., "it fits," "this overlaps") they used to talk about their comparisons. He would use these words to induce the statement of general properties of area as they justified their comparisons. Against the background of that initial division of labor, Earl, Didi, and Gina came across other, more complicated strategies that required new negotiations: directing attention to the elements of the triangle area formula and directing attention to quantitative comparisons between dimensions of triangles.

A key element in the formulation of Didi's conjecture was the use of language elements that belong to the area formula. The words base and height came to be used in segment 6 when Gina reported to Earl that they thought triangle D was bigger than triangle C because "the length" of D was bigger than the length of C. In response to Earl's question "What does it mean to talk about the length of a triangle?" Didi added that "though C is longer that way than D is, D is like blown [makes gestures that suggest a much longer other side]" (see Figure 1). Earl intervened, "What I hear you saying Didi is that C has a slightly bigger base or width, right? But D has a much bigger height. So it feels that it should be bigger, right?"

Thus, attention to the elements in the formula (base, height) had come in the context of negotiating how to reconcile an observation about those two particular triangles with a more general issue—that triangles extend in two dimensions. Earl's question had the force of indicating that "length of a triangle" is an ambiguous expression, an ambiguity that Didi meant to resolve by calling attention to a second dimension.

But whereas Didi's comment correctly brought attention to the bidimensionality of area and perhaps provided an ad hoc justification for that particular comparison, her comment was not completely helpful for Earl to bring attention to the properties of area. Because Didi was comparing how those two triangles differed in terms of two of their sides—one slightly longer, one much longer—for Earl to accept that justification would have involved contravening what he knew to be true about area. It is understandable that Earl would then say, "Remember now, what do we use to talk about . . . the height of a triangle? . . . We are not actually measuring this length [Earl points at a slanted side], right? We would be measuring from the point to a right angle at the base."

Earl thus brought in base and height to introduce precise language that could mold Didi's reason into something closer to being mathematically acceptable. His suggestion entailed a trade. He accepted Didi's observation that to compare two triangles according to how tall they are, one must balance that comparison by also paying attention to how wide the triangles are. He gave back language (base, height) for talking about these two-dimensional comparisons and held students responsible for drawing perpendicular heights should they want to use this kind of justification. Thus, the elements of the
area formula entered the discussion not as numbers to be plugged into a calculation but as the language to designate the dimensions of triangles that are relevant in area comparisons. They did not seem to be perceived by students as contravening the constraint.

A second key element in Didi's conjecture was the use of ratios to quantify the relationships between the dimensions of triangles ("twice," "half"). This way of quantifying relationships appeared first in segment 8: While comparing triangles $E$ and $F$, Gina said $E$ was bigger by virtue of having greater base and height. Upon Earl's question of whether she had measured in order to be able to determine it had greater base and height, Gina responded that she had used the compass. She added later that "you can tell it has a greater base and a greater height because... look it is almost three fourths of the other side." Later, in segment 10, Gina called Earl to ask about triangles $D$ and $E$: "$E$ has a greater width than $D$ does but $D$ has a greater height than $E$ does, so how do you tell which one is greater." The following dialogue ensued:

_Earl:_ Can you... _measure..._ [elongates vowels and sounds doubtful while uttering word]

_Gina:_ What are they asking? What are they asking we do with this?

_Earl:_ [over Gina's question] Can you measure the heights and bases to see like for example if... If the... if this height is like three times... if this base is like three times that one... and the heights are double, that kind of thing...

_Didi:_ We would have to use this [grabs the compass].

_Earl:_ Well, you're allowed to use the ruler. I'm not saying you can't use the ruler.

_Gina:_ But you don't have to use the ruler, you can use the compass.

_Earl:_ Well, compass will tell you which one is bigger but I don't know if it will tell you. [Didi: How many times, ...] Will it?... The ruler... I didn't say not to use the ruler, I said not to... try not to use the area formula. In the meantime, Didi has been comparing bases with her compass—opening to capture a side of $E$ and fitting that opening twice on $D$.]

_Gina:_ By twice, right?

_Didi:_ [overlapping Gina] This [Es base] fits twice [in Ds base].

_Gina:_ $D$ is twice the width, the base.

When Earl came back to Didi and Gina in segment 12, they had already been trying some things to compare the altitudes. Using an arbitrary opening of the compass, Didi had discovered that such an opening fit five times along the "height" of $D$ and about two and one third times along the "height" of $E$. Yet, they seemed to be taking the heights not along perpendicular segments $HL$ and $PO$, but rather comparing sides $ML$ and $MO$ (see Figure 3). Earl
insisted that they should use the ruler and the plastic right triangle to try to consider the actual altitudes; this insistence was met again with expressions of frustration from both students.

There are two outcomes of these negotiations from segments 8, 10, and 12 that matter to our understanding of how Didi's conjecture was produced and treated. First, the notion that quantifying the relationship between dimensions was useful emerged initially from Gina (in segment 8) as a way to defend her comparison. Earl brought it up again to facilitate their gathering of resources for another comparison. In doing that, he imposed on them the responsibility to discover ratios between segments, as a way to push the discussion into making finer two-dimensional comparisons. Second, to make his quantifying suggestion viable, Earl conceded a drastic change in his expectations, from expecting them not to use the ruler (segment 6) to suggesting they use the ruler (segment 10). Gina's response to his suggestion to measure and Earl's reference to the statement of the task pointed to the fact that the concession of using the ruler could be construed as a breach of the didactical contract. Earl's emphasis that the constraint was not on measurement but on the use of the area formula can thus be understood as an attempt to prevent that breach.

The foregoing description of segments prior to the emergence of Didi's conjecture shows how the elements of an emergent story line were coming to play in the context of Didi's and Gina's work comparing triangles. At a time when the particular area comparisons they were making could still be justified on empirical grounds, the elements that would permit the formulation of a reasonable (but not evident) claim were already finding their way in as by-products of the negotiation of the didactical contract.

**Second moment: The emergence of Didi's conjecture**

At the end of segment 12, Earl had left Didi and Gina measuring the altitudes for D and E with a ruler. As he came back in segment 15, Gina asked him to mediate in a disagreement over a claim that Didi was making. Gina said, "She thinks that makes it equal but I don't... because this base [points at triangle E] is greater and this height [points at triangle D] is greater, how can they be equal?"

The previous section showed how the information on which Didi based her claim had found its way into the interaction. What had remained unclear was how that information should be combined to produce a comparison. Thus, for instance, when they had compared triangles C and D (segment 6) they had spoken about bases and heights as a way to justify a comparison that was perceived intuitively. In contrast, whether D or E was bigger was not so evident on perceptual grounds—and thus what was needed was not simply a justification for an answer but also a way of reasoning that would allow them to figure out the answer. Comparing D and E required them to find a way to combine the quantitative relationships of bases and heights they already knew. To find a way to do so was crucial at this point because, as they would confess later to Earl, they had also tried (without success) to
make one of the triangles fit inside the other. Didi's claim that the triangles should be equal had no perceptual support (they looked so different, as Gina said, "how could they be equal?"). Didi's conclusion, however, was based on an implicit discovery of hers. The area formula could be used to collect the quantitative relationships they had found among the dimensions of the triangles and to combine them in order to anticipate the relationship between their areas: "It cancels ... if you do ... base times height."

Didi's idea seemed to properly build on previous issues that they had been working on (in segments 6, 8, 10, and 12)—notably on the value of comparing bases and heights and on establishing those comparisons in terms of ratios. But, however connected with previous work Didi's idea might have been, Gina's reaction was pertinent. Her statement, "You are not allowed to use the [area] formula," gave warning that Didi's explanation might breach the norms that Earl had identified in segment 10 ("I didn't say don't use the ruler, I said try not to use the area formula"). This is an important moment for the negotiation of the didactical contract. On the one hand, previous negotiations (particularly the encouragement to use the ruler) had helped the development of Didi's conjecture—a conjecture that was certainly among the important new mathematical ideas that the task permitted to develop. But on the other hand, Didi's conjecture openly violated the official conditions for the execution of the task. Didi was saying something smart and hoped for, but to some extent unacceptable.

Such a breach in the norms that had made the undertaking of the task possible might require special work—a new negotiation of the contract for the given task, or perhaps a new task. It is conceivable, for example, that Earl and his students might have engaged in a discussion that led them to agree that Didi was not really using the area formula in the way alluded to by the constraint when the constraint was issued. Such discussion might have led them to identify a new way of making area comparisons. But Gina did not seem to be ready to go down that road. Earl's response came to mediate that disagreement by focusing on Didi's claim that the triangles were equal in area:

I have an idea ... well you can ... you could use it ... should check. In fact, this might be a good time to ... to check Didi's hypothesis here. But another thing you could do is try comparing it a different way if ... have any of the strategies you guys have used prior to this ... would [there] be a way to check the comparison of D and B another way? See if she is right that they are equal?

And after they verified that break-and-make would not yield any conclusive comparison, he suggested:

I think this would probably be a reasonable time to try the formula and see if it confirms Didi's conjecture. ... [Gina takes the lead organizing who will calculate which area.] But before you do that, why don't you write it out, that they are equal. And basically your conjecture is that if one has double the base and the other has double the height then they should be equal.
Earl’s intervention managed to maintain some sense of stability that deserves further comment. First, Didi’s claim (that D and E are equal in area) was kept as a possibility. The implication seemed to be that because they already held as true what they presumed about ratios between dimensions, if they also got independent verification that D and E were equal, they would have reason to believe in Didi’s conjecture. Second, Earl’s intervention seemed to succeed in shifting attention from the breach in the didactical contract that Gina had detected. The area formula that Didi had used in a novel way to respond to Earl’s request to “explain [her] thinking” would now be used in the familiar way, to provide verification. Yet, the “reason” that would get written in the worksheet, should the claim be true, would only mention the ratios, not the area formula. Third, as the constraint had just been to “minimize the use of the formula,” Earl did not seem to lose so much ground in suggesting that “this is a good time to use it, to check Didi’s hypothesis.” Furthermore, the suggestion to use the formula to verify without acknowledging that Didi’s use was different served to level the playing field between the two students, to leave them on equal footing. Earl could therefore keep the students focused on the central task of comparing triangles and also maintain parallel attention to Didi’s emerging conjecture. In the next section, I examine how the proceeds of the “verification” affected the development of this parallel story line.

Third moment: Dealing with Didi’s conjecture

Gina’s objection to Didi’s novel use of the formula had had the effect of maintaining the attention on the original task of comparing triangles. At the same time, the discourse indicates that Earl’s interest in Didi’s conjecture was based on more than his possible desire to encourage Didi or the presumable usefulness of the conjecture to compare triangles D and E. In segment 17 Earl invested considerable attention in discussing whether Didi’s conjecture was reasonable, in addition to knowing whether the triangles were actually equal. This supports the assertion that, for Earl, Didi’s conjecture was important on its own terms.

When Earl came back to hear what Didi and Gina had found out from their calculations of areas, and they reported that the triangles were not equal, Earl explained why such a result addressed the comparison task without falsifying Didi’s conjecture. And he suggested attention to the conjecture on its own right.

_Earl:_ When you measured the two heights, were they double each other?

_Didi:_ Almost.

_Earl:_ Almost. . . . So if they are only almost double . . . maybe you should pretend they are double and calculate to see if your hypothesis is right.

But Earl did not press them to follow his advice. He rather came back to the comparison task, suggesting that more careful measurement might yield...
the result they expected. After getting himself involved in another try at measuring the heights carefully, Earl pointed out that “maybe it was not true that it was double.” Earl then took back his earlier suggestion, “But if it is double . . . try this as 9 and this is 10 and a half, and pretend this is .21 and this is 4 and a half and do the calculation to see if Didi’s conjecture is reasonable.”

As the students found out that with those hypothetical dimensions the areas would be the same, Earl interpreted the result as suggesting “that Didi’s conjecture could be right,” though he qualified it by saying that they had not proved it. To Gina’s relief he added, “That doesn’t mean these two triangles are the same.”

As Earl left the group, he suggested that they continue working on other triangle comparisons. One can see in the sequence of moves back and forth between the conjecture (that these ratios between dimensions imply equal areas) and the claim (that these triangles are equal) how Earl could not quite unite the two stories. This observation contrasts in a marked way with what had been the case with previous comparisons in which he had been able to take students’ descriptions of how they had compared figures and use them to formulate more general propositions about area. In the comparisons that followed segment 17, measurement and calculation—that is, the familiar use of the area formula—became the norm. Earl placed much more emphasis on drawing heights that were exactly perpendicular, measuring bases and heights accurately, and computing areas to verify all comparisons. The novel, algebraic way of using the area formula to determine ratios between areas never came back to the fore.

Earl took up Didi’s conjecture near the end of the lesson, in the context of revisiting with the whole class all of the comparisons that they had made. In this whole-class discussion they found out that the other students, Mina and Tara, (a) had not developed anything similar to Didi’s conjecture and (b) had actually found $E$ to be bigger than $D$ for reasons they could not articulate well. In segment 31, Earl asked Didi to share her conjecture as something interesting in its own right, and he illustrated it to the class using particular numbers for bases (4 and 8) and heights (6 and 3). The issue of whether $D$ or $E$ was bigger was left to the following day but was not pursued then or anytime thereafter.

**Actions of the Teacher in the Negotiation of the Contract That Need Explanation**

In the foregoing section, I tracked those of Earl’s actions that were crucial in enabling and shaping the emergence of Didi’s conjecture and argument in the context of the ranking triangles task. I have shown that Didi’s claim that both triangles were equal was important insofar as it was a claim based on mathematical reasoning rather than empirical evidence. Her possible reasons to assert that claim were more important than the claim per se, insofar as the way her conjecture could be argued entailed using the area formula in an algebraic way—a novel way for these students.
The chances for the ideas implicit in Didi’s actions to come to the fore hinged on at least three conditions. First, the product of the task had to be directed toward examining the intellectual tools for making area comparisons and away from merely making those comparisons. Second, the triangles as resources for the work had to be represented primarily in terms of symbols and their relationships, rather than identified holistically with the cardstock figures. Third, the constraint on the operations to be carried out had to be conceived more as a covert opportunity (use it to find a way around using the area formula) than as an overt limitation (insist that a solution should be found without using it at all).

Making room for Didi’s conjecture as students worked in this task thus required negotiating the didactical contract; some of the outcomes of this negotiation have been described, and they point to the intrinsically difficult nature of the work of the teacher. First, we saw that whereas Earl succeeded in bringing some attention to Didi’s conjecture, he could not negotiate for it a central place in the lesson or bring forth other conjectures like it—the main focus of students’ work continued to be comparing areas of triangles. Second, we saw that whereas Earl succeeded in bringing in the language of base and height to describe triangles, the objects of the task continued to be the empirical, cardstock triangles. Third, we saw that whereas Earl succeeded in relativizing the constraint, the area formula continued to be perceived as a device to crunch numbers rather than as a device to express quantitative relationships.

Three Tensions in Managing Novel Tasks

In this section, I show how the three tensions described earlier help provide a plausible explanation for why Earl’s actions shaped the mathematical ideas emerging from this task in the way described above. Evidence that supports using these tensions to explain Earl’s actions comes from his teaching of the lesson as well as from the comments he made in interviews before and after the lesson.

Where to Direct Students’ Activity When a New Idea Comes Up

Whereas a novel task may be announced to students as oriented toward obtaining a product, the most important “products” aimed at by the task may be hidden from the students’ view when they start their work. These products—in particular, the ideas that students may come up with—may be foreseeable by the teacher as he or she chooses a task. But because those ideas are meaningful in the context of students’ work on the task, and perhaps even embodied in students’ actions, it may be difficult for a teacher to separate them from the task that provided the context for their emergence. A mathematically educated observer may be able to distinguish these ideas and talk about them, yet they may be “ineffable” (Ball & Bass, 2000) or “fragile” (Brousseau & Otte, 1991) in the interactions between teacher and student. The chances for those ideas to live in the classroom may hinge on
the difference they make in specific situations (Bateson, 1971, pp. 454–471), and special work will need to be done if they are to be formulated in general (Balacheff, 1990).

Hence, as a teacher manages students’ engagement with a novel task, he or she may be disposed to supporting two opposing recommendations. On the one hand, a teacher may feel obliged to maintain students’ attention on the announced product of the assigned task, perhaps building up experiences that could later be used to formulate new ideas. On the other hand, a teacher may feel compelled to redirect the activity so as to seize the opportunity to develop an emerging new idea implicit in students’ work, avoiding the risk of losing the singularity of the new idea in the continuum of experience (Mason, 1999). Rather than offering a clear choice, these recommendations create a tension, which is illustrated in the episode under consideration by Earl’s need to pay attention at the same time to the comparison of triangles and to Didi’s conjecture and argument.

At various moments in the interviews, Earl had expressed a commitment to value individual students’ thinking. That by itself might explain why he would pay attention to whatever Didi dared to offer. But beyond that, and more important insofar as Didi’s conjecture called attention to ratios and the multiplicative relationships embedded in the notion of area, her conjecture fell within the range of things that Earl was tuned to recognize as a legitimate outcome of the task assigned, because it had been anticipated as we planned the lesson. The task had been issued in order to develop awareness of some of the very things that Didi’s conjecture was pointing at. To let Didi’s conjecture pass would not be a way for him to keep his responsibilities to the subject matter, no matter that the conjecture had failed to produce a useful response to the task. Rather, the fact that the comparison between triangles $D$ and $E$ had resisted the application of the conjecture might perhaps suggest to Earl that the task needed to be adjusted so as to enable students to focus on what was important. Earl could have used his position in the didactical contract (the presumption that students know he is responsible for their opportunities to learn) to warrant changes in the direction of their work. On that assumption, it would have been fitting for him to direct attention to Didi’s conjecture and argument and the properties of area they addressed.

Yet, Earl’s disposition to recognize the value of Didi’s conjecture likely had to be moderated with other dispositions that, based on Earl’s comments in the interviews, were also activated in this lesson. He valued maintaining a working environment that allowed fair opportunities for all students to know and do mathematics. Important in that consideration was Earl’s recognition that Didi’s conjecture and argument were highly underdeveloped and somewhat fortuitous. It was a “feeling” that Didi had had in the context of a comparison but one that she had not been able to articulate well to her partner Gina. The emergence of the conjecture was contingent on how the two students had chosen to look at the two triangles to such an extent that it would have been difficult for Earl to help the other pair of students to reproduce the finding independently. In addition to that, Didi’s reasoning had
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appeared to Gina as an idiosyncratic interpretation of the task. It would have been difficult for Earl to argue publicly that his plan had included directing all students' work toward exploring something so seemingly accidental and private as Didi's conjecture and its justification.

Furthermore, while as the teacher Earl was entitled to change directions based on the ideas that emerged from the work, it does not seem as though he could afford to allow students to perceive the ranking triangles task as a mere motivational instrument. Rather, he had to keep in mind that the task itself had substantive value in at least two senses. The task created meaning for the ideas at stake, and it identified a piece of the students' share of labor in the study of area. A change in the task might critically alter students' expectations vis-à-vis recognizing what was important to pay attention to or dedicate effort to. Earl could probably foresee that if he were to shift attention away from the task before its end, this could jeopardize students' engagement in future tasks. And whereas the ideas emerging from the work might have been mathematically more important than the work at hand, they were still important because of their instrumental value.

To maintain a "continuity" and keep focus on the task, rather than to impose a "rupture" and change the task, seemed thus to be the preferred option in this particular circumstance. Faced with the need to deal with Didi's conjecture, Earl managed a way for Didi and Gina first, and the whole class later, to pay some attention to it without shifting attention to a new task. In doing that, he missed the opportunity to bring to the fore other entailments of the multiplicative structure of the area formula, but he maintained students' active involvement in the work, keeping a reasonable degree of homogeneity as they worked. I would like to suggest that this tension between negotiating how to assimilate new ideas into an existing task and negotiating how to adapt the task to better serve these new ideas is always present when novel tasks are used in teaching. The enactment of a task provides meaning for emerging ideas. To attend to those ideas seems crucial in developing new knowledge. However, it is extremely complicated for a teacher to direct attention to new knowledge without taking away from that new knowledge part of its meaning which depends on students' actions in the context of the task.

How to Represent the Mathematical Objects Involved in a Task

Any task involves resources that students may use in their work. These resources include representations of mathematical objects crucial for students to use in developing new ideas. Representations that do not openly disclose what might be relevant for a given task afford students an opportunity to engage in a process of inquiry where they may choose what to pay attention to based on the ideas they consider relevant. Inversely, representations that unambiguously point at what is relevant about those mathematical objects are useful in increasing the chances that students do get to work with the ideas intended by the teacher. To be able to develop and value abstract mathematical ideas, students need to experience abstraction as a
meaningful choice in organizing their actions in context (Herschkowitz et al., 2001). But rich experiences do not guarantee that abstractions will be developed. How to represent mathematical objects is thus a difficult aspect of the teacher's work in managing a novel task.

As a teacher manages students' engagement with a novel task, he or she may be disposed to supporting two opposing actions. On the one hand, a teacher may feel pressed to identify which features of the representations used in the task are relevant to the mathematical ideas that he or she wants students to investigate. On the other hand, the teacher may feel inclined to honor the richness of information implicit in those representations, leaving for students more opportunity to choose how to use them, to mathematize. These recommendations may create a tension. This tension is illustrated in the episode under consideration by Earl's struggle with the distinction between the abstract features of those triangles that were relevant to Didi's conjecture and the concrete, specific features of the actual cardstock triangles whose comparison originated Didi's conjecture. The tension is particularly manifest in Earl's contention that "there [were] two issues"—whether the triangles are equal in area and whether Didi's conjecture is true—both competing with each other for center stage and yet warranted by one and the same triangle comparison.

Until Didi and Gina started working on triangles $D$ and $E$, the distinction between concrete cardstock triangles and abstract mathematical triangles had been immaterial—the lack of differentiation had actually been helpful in eliciting properties of area. The actual triangles acted as proxies for area, allowing students to work in an environment where they could directly experience the concept of area. For example, as triangles could actually be juxtaposed and superimposed, students could experience several ways of comparing areas using geometry; their accounts of those comparisons could then be used without apparent problems to state abstract properties of area. However, if Didi's conjecture could emerge, it was because at one point in time she took the chance of detaching her thinking about area from the appearance of the particular triangles she was comparing. But it was difficult for Earl to manage the shifting of attention from the concrete cardstock triangles to the abstract, mathematical representation of triangles where Didi's conjecture could be treated.

It was important for Earl to maintain attention on the cardstock triangles. Indeed, the concrete triangles had already supported a variety of approaches that increased students' active involvement, something he valued and wanted to improve on. In addition, as Earl had indicated, a fundamental goal of his for this lesson was to have students understand that the concept of area is more general than the application of a formula to calculate the area. The actual set of figures was a messy environment that, however, afforded students access to the basic properties of area. The facility to manipulate the figures and the variety of resources provided in the toolbox supported the development of those properties without giving them away. In addition to that, the fact that neither ratios nor numbers had been given at the outset was what
made Didi's conjecture and claim about triangles $D$ and $E$ so meaningful—a difficult comparison within that messy cardstock environment had triggered a choice to abstract and mathematize.

However, because Earl was teaching mathematics, it was also important for him to ensure that attention did not remain with the cardstock figures themselves. He rather needed his students to use the figures as mere embodiments to think about triangles in general and their areas. At times students might make comparisons that might be true for the actual triangles involved but justify them using reasons that were not mathematically acceptable, and in those cases Earl seemed to be disposed to call attention to the mathematical features that should be considered. At other times the openness of the representations actually seemed to stand in students' way of coming up with important ideas. For example, the options to cut or measure figures introduced additional uncertainty regarding the "true" measure of objects.

The need for Earl to help students interpret their measurements of triangles $D$ and $E$ shows how his dispositions toward representations entered into conflict. It seems to have been hard to reconcile the perception that measures taken with a ruler gave exact information about these triangles with the notion that such exactitude might include more error than what could be accounted for with their current knowledge of geometry. Earl's decisions to accept that triangles $D$ and $E$ were different and to salvage Didi's conjecture by providing an ad hoc example using numbers were ways for him to capitalize on the different kinds of work that students had been doing. In doing that he avoided having to deal with the fact that those messy representations and their contradictory proceeds had actually been the product of a choice in the design of the task. To push students to represent the problem algebraically after the lengths had been measured would not have helped much in showing how relevant Didi's conjecture was. But the suggestion that they compute with actual numbers that stood in the desired ratios did not only point to a way of verifying Didi's conjecture, it also provided an example of what measures they would have needed for her claim to be true. He handled the divergence between an empirical truth about concrete objects and a plausible truth about abstract triangles by constructing an experience closer to the mathematical theory so that the theory could be "seen through" the experience. At the same time, he protected the way of building mathematical theory from being abused by adding the caveat that "they had not proved it."

This tension between attending to particulars of concrete representations and to the generalities of abstract mathematical objects goes beyond the specifics of how Earl managed his students' work on Didi's conjecture and claim. The need to find out how to represent mathematical objects exhibits how it is necessary and difficult for a teacher to manage the flow of information between representations and mathematical ideas. To the extent that novel tasks are contexts in which students work with representations in the pursuit of new knowledge, this tension is a useful way of looking at the management of knowledge in any novel task.
How to Elicit Students’ Conceptual Actions That Are Instrumental for the Task

Any task involves *operations* that students undertake in order to arrive at the product. As suggested earlier, the determination of which intellectual operations are instrumental in completing a task is an important part of what students might gain from their work on a novel task. Directions and constraints for a task are normative in that they establish expectations regarding what may and may not be done. Directions and constraints are also productive in that they enable students to come up with those intellectual operations by demarcating the relevant intellectual terrain of work. Thus, directions and constraints are not only given for students to abide by them, they are also available for students to think *with* and *against* them. Because they do those two things, directions and constraints can be ambiguous, sources of information for students. Managing how these directions and constraints enable students to come up with operations to complete a task is a challenge to the work of the teacher.

As a teacher manages students’ engagement with a novel task, two opposing recommendations may shape his or her action in regard to the interpretation of directions and constraints. On the one hand, a teacher may feel pressed to reduce the ambiguity of directions and constraints by specifying to students *what are the intended interpretations*. On the other hand, a teacher may feel inclined to *maintain such ambiguity unresolved* as an invitation for students to invest themselves in deciding what makes sense for them to do. These recommendations may create a tension, illustrated in Earl’s struggle with how to enable students to realize the difference between two possible ways of using the triangle area formula.

As has been narrated, Earl at times insisted that students should not use the area formula to calculate areas, whereas at other times he expected them to use what they knew about the area formula to compare triangles. As they came to Didi’s conjecture, he ran into the problem of deciding whether or not Didi had violated the constraint in her way of justifying her conjecture. Didi had obviously used the area formula, though not to calculate areas but to discover a relationship among areas. To come up with her conjecture, Didi had dared to *think with* the formula without using it in the way alluded to by the constraint. Yet, Gina’s objection to Didi’s move seems to have been so important to Earl that it warranted withdrawing the constraint thereafter at the expense of forfeiting the opportunity to attend to Didi’s algebraic way of using the area formula. This action of Earl’s can be understood using the tension outlined above.

It had been valuable for Earl to keep the constraint ambiguous. The constraint referred to calculating actual areas only implicitly; it did not even suggest the possibility that other uses of the formula existed. The challenge to abide by the constraint thus gave students impetus to retrieve and make explicit other basic area properties. Had Earl disclosed the legality of using the area formula to express relationships among quantities, “Didi’s conjecture”
Using Novel Tasks in Teaching Mathematics

might have appeared, but it would have represented less of an intellectual accomplishment. Thus, the possibility for Earl to observe the emergence of Didi's conjecture and interpret it as a meaningful piece of mathematical thinking was concomitant with maintaining the constraint as ambiguous, a productive ambiguity. Since all students had had school experiences with algebraic expressions in eighth grade, it was not unreasonable to hope that they might choose to invest their knowledge of algebra in this task.

Therein lies the tension. Had all students known they were allowed to use algebra, they would have likely done it. The ambiguity of the constraint also made Gina's objection meaningful in that it might have expressed to Earl an essential unfairness in the task. Should Earl have not explicitly asked them not to use the formula, Gina might have also come up with the same idea. Thus, for Earl to introduce the constraint and at the same time value an algebraic use of the formula in public might have seemed like a betrayal to those who followed his directions. Gina's complaint against Didi's use of the formula indicated to Earl a fundamental inequity that could be construed as springing from normative deficiencies in the statement of the task, an inequity that, in view of Earl's explicit commitment to all students, he likely felt responsible to repair. To have distinguished at the outset between two ways of using the area formula, by saying that they might use the area formula to express an unknown quantity or by giving specific examples on how to use the area formula to express quantities, would have eliminated the ambiguity.

Earl's follow-up to Gina's objection, indicating that they could use the area formula to verify numerical equality (and insisting that they verify other comparisons thereafter), can be understood as a compromise between the two opposing tendencies. In legitimizing Didi's thinking, Earl turned the constraint into a preference that was acceptable to disregard. To prevent Gina's disappointment over the ambiguous nature of the constraint, Earl relinquished bringing up the difference between the new, algebraic use of the area formula present in Didi's thinking and the old, computational use. The expectation that the truth of Didi's conjecture could nevertheless be seen through the particular numbers being used (21 and 4.5; 10.5 and 9) seemed to help him reach that compromise. The class could thus be on the same page in relation to the specific fact stated in Didi's conjecture. And Earl might hope that they would make subsequent comparisons using the multiplicative structure of the formula to operate with ratios seen through the numbers.

This tension between eliminating and maintaining ambiguity over directions and constraints for a task goes beyond the specifics of how Earl managed his students' work on the ranking triangles task. The tension illustrates how it is necessary and difficult for a teacher to reconcile the productive and normative roles of directions and constraints regarding elicitation of the conceptual actions expected of students. To the extent that novel tasks are contexts in which students are developing awareness of new ideas against the background of old ideas, this tension is a useful way of looking at the management of knowledge in any novel task.
Conclusion: How Do These Tensions Help Us Understand Teaching?

"When novel work is being done," states Walter Doyle (1988, p. 174), "activity flow is slow and bumpy." Also, "novel work stretches the limits of classroom management and intensifies the complexity of the teacher's task of orchestrating classroom events" (p. 174). The argument presented in this article goes deeper in that direction, suggesting that novel tasks intensify the complexity of a teacher's management of the development of knowledge. The argument acknowledges that novel tasks can be used as contexts for students' development of new ideas and inspects what teachers do to negotiate the conditions of work in such contexts. It is argued that managing knowledge development in such contexts is difficult for teachers as a result of three possible tensions centrally related to the structure of academic tasks.

The existence of these tensions follows from the postulate that teacher and students constantly negotiate a didactical contract that permits them to interact around the work they do as well as interpret such interactions as accomplishing their mutual, global obligations in regard to the teaching and learning of mathematics. Such negotiation affects the shape that mathematical ideas take in the classroom. My job has been to describe the shape taken by ideas of area in Earl's class, as well as the negotiation of the didactical contract that allowed Earl and his students to work on those ideas, and to examine how the latter affects the former. I have investigated what a teacher does in that negotiation when the work at hand is novel and suggested that the way in which a teacher's actions affect the nature of the mathematical ideas at play may be explained by the hypothesis that he or she has to manage three subject-specific tensions. These tensions concern the product of, the resources for, and the operations for a task. First, in negotiating the direction of students' activity, a teacher must manage a tension between maintaining attention on the explicit product of the assigned task and redirecting attention to emerging new ideas implicit in students' work. Second, in negotiating the use of representations of mathematical ideas, a teacher must manage a tension between maintaining attention on the informational richness of the concrete representations used and directing attention to the features of those representations that refer to the ideas under study. Third, in negotiating the conceptual operations that students need to undertake, a teacher must manage a tension between maintaining a productive ambiguity regarding what is relevant to do and specifying unambiguous norms regarding what is required to do. While managing those tensions, a teacher's actions shape in substantial ways the mathematical ideas that students have the opportunity to learn about.

Insofar as these tensions have been useful in interpreting the management of task and knowledge in one particular case, no empirical claim is made regarding the frequency or strength of their presence. Rather, I suggest that these tensions are useful theoretical constructs that can be used to understand how teachers manage novel tasks. In particular, they can be used
Using Novel Tasks in Teaching Mathematics

as heuristics to explain why sometimes teachers play a role in changing the cognitive demands of mathematical tasks (Stein et al., 1996) or why they tend to assign more familiar than novel tasks (Doyle, 1983). These tensions are also useful guides for inquiring into teachers' use of curriculum materials. Specifically, as they propose "worthwhile mathematical tasks" (National Council of Teachers of Mathematics, 1991), curriculum designers need to contend with the question of which of those tasks practitioners may consider viable instruments with which to teach. Inquiry that allows for reasoned engineering of curriculum materials can use the hypothetical tensions I propose to systematically generate alternative modifications of proposed tasks and to explain the variance in teachers' assessments of the viability of possible tasks.

It is probably true that teachers' personal knowledge, commitments, and beliefs about mathematics and its instruction affect their behavior when managing the development of knowledge (Thompson, 1992). It is also likely that the crucial responsibilities that all teachers have with respect to the subject matter and their students contribute to the development, organization, and activation of such personal knowledge, commitments, and beliefs. From a theoretical standpoint, the argument presented here is an example of how phenomena concerning the mathematics at play in classroom interactions can be explained in terms of the characteristics of the practice (the action in context) of teacher and students (see also Arsac et al., 1992; Ball & Bass, 2000) rather than in terms of who the practitioner is or what he or she knows.

Future steps in theoretical development must include ways of reconciling these practice-based explanations with explanations based on the beliefs, commitments, and knowledge that practitioners bring with them (see Herbst & Chazan, in press). From a no less important practical perspective, the argument discourages analyzing a teacher's work in terms of assets and deficits: If teachers do things that seem odd to a discipline-oriented observer, that is a sign to be understood through an internal critique before being a course of action to be judged. Indeed, the argument suggests that in the work of developing ideas with his or her students in class, the responsibilities that a teacher must attend to play out very differently than those of a text author leading his or her readers in a narration of such development. A mathematics teacher and his or her class create an original mathematical performance as their interactions unfold in context. As in the performing arts, their performance is amenable to a mathematical critique. And like in the performing arts, such a critique cannot be reduced to a critique of the script on which the performance is based or of the mental states of the individuals who act in the performance. Specific language is needed to describe and understand the qualities of classroom mathematical performances—the three tensions described serve that purpose.

The argument presented here stays within a teacher's management of novel mathematical tasks. Yet, it invites questioning whether managing novel tasks is different across subjects of study. The present argument makes it clear that the three tensions exist by virtue of the fact that novel tasks are not only contexts for work but also vehicles for teaching about ideas. However,
the teaching of any discipline requires a teacher to negotiate with students interactions with and about the objects of study. Could similar tensions help understand the teaching of subjects other than mathematics if similarly detailed attention were paid to the specific ideas at play? Perhaps these tensions play out similarly in disciplines that are organized around general, abstract concepts (such as physics or biology) but not so similarly in disciplines where particular events or cases play a central role in knowledge organization (such as literature or history). Or perhaps what distinguishes how these tensions play out across disciplines can be related to the different cognitive mechanisms with which children develop specific disciplinary ideas. I suggest that the tensions described above can be useful heuristics for comparative studies on how the work of teaching subject matter is related to the knowledge that students can develop and use in classrooms.

It is a subject of further inquiry whether and how this language and logic of tensions can be helpful to teachers in doing their work. The three tensions analyzed here may be useful heuristics for teachers to search for management challenges that particular tasks may present to them and anticipate how to cope with those challenges; the tensions should not be taken as excuses to avoid teaching with novel tasks but as tools to figure out what is at stake when doing it. The argument also informs the work of teacher educators and developers in promoting sophisticated ways of blending improvement with empathy. The three tensions caution against simply seeing teachers as people who need to know what to do to improve their practice; they recommend that we look empathetically into the actions of teaching as those of people who must contend with complex, neither good or bad, demands placed on them by their position vis-à-vis students and subject matter.

**APPENDIX**

**Summary of Lesson Segments**

<table>
<thead>
<tr>
<th>Segment</th>
<th>Duration</th>
<th>Succinct description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1m 20s</td>
<td>Earl stands at his desk revising his notes while students, sitting in pairs, are talking about things unrelated to class. He asks students to not pay attention to the cameras.</td>
</tr>
<tr>
<td>2</td>
<td>30s</td>
<td>Earl walks toward students' tables and reveals the contents of the toolboxes for students.</td>
</tr>
<tr>
<td>3</td>
<td>4m</td>
<td>Standing at the board, Earl reviews with students the concept of area and the formula for area of a triangle. He then introduces the ranking triangles task.</td>
</tr>
<tr>
<td>4</td>
<td>1m</td>
<td>Earl approaches each group of students to give them the worksheet while explaining to all how to use it.</td>
</tr>
<tr>
<td>5</td>
<td>2m</td>
<td>Earl approaches Mina and Tara to ask for questions. They work on a comparison involving two triangles, one of which has sides that are longer than all sides of the other.</td>
</tr>
<tr>
<td>6</td>
<td>2m</td>
<td>Earl approaches Didi and Gina, who are comparing triangles (E) and (D). As they do that comparison, they start to use base and height to speak about the two dimensions.</td>
</tr>
<tr>
<td>Segment</td>
<td>Duration</td>
<td>Succinct description</td>
</tr>
<tr>
<td>---------</td>
<td>----------</td>
<td>----------------------</td>
</tr>
<tr>
<td>7</td>
<td>3m</td>
<td>Earl works with Mina and Tara on a comparison that leads to the idea of comparing by inclusion.</td>
</tr>
<tr>
<td>8</td>
<td>2m 30s</td>
<td>Earl works with Didi and Gina on a comparison, and they make the claim that if a triangle has bigger height and bigger base, its area is bigger. They also realize that they could compare by inclusion.</td>
</tr>
<tr>
<td>9</td>
<td>1m</td>
<td>Earl works with Mina and Tara on a comparison that aims at connecting angle size and areas.</td>
</tr>
<tr>
<td>10</td>
<td>1m 15s</td>
<td>Earl discusses with Didi and Gina various options to handle comparing triangles D and E.</td>
</tr>
<tr>
<td>11</td>
<td>1m</td>
<td>Mina and Tara call Earl, undecided on something inaudible. Earl offers them scissors.</td>
</tr>
<tr>
<td>12</td>
<td>2m</td>
<td>Earl comes to Didi and Gina's table. He suggests that they measure and make heights perpendicular to be precise as they make claims about the ratios among dimensions of D and E.</td>
</tr>
<tr>
<td>13</td>
<td>1m 30s</td>
<td>Earl looks at his papers on his desk.</td>
</tr>
<tr>
<td>14</td>
<td>1m 10s</td>
<td>Earl visits with Mina and Tara, who have justified a comparison by using the notion of inclusion.</td>
</tr>
<tr>
<td>15</td>
<td>3m</td>
<td>Earl responds to a call from Gina. Didi claims and argues that triangles D and E should be equal in area.</td>
</tr>
<tr>
<td>16</td>
<td>3m</td>
<td>Earl visits with Mina and Tara to discuss various comparisons they have made. They mention the idea of comparing triangle areas by comparing bases and comparing heights.</td>
</tr>
<tr>
<td>17</td>
<td>4m 30s</td>
<td>Earl comes back to Didi and Gina's table to discuss results from careful measurement of Ds and Es dimensions and the calculation of their areas.</td>
</tr>
<tr>
<td>18</td>
<td>1m</td>
<td>Earl works with Mina and Tara comparing bases and heights of two triangles.</td>
</tr>
<tr>
<td>19</td>
<td>1m</td>
<td>Earl is at his desk looking at his papers. Then he announces to each group of students that they could start using the information collected to make a rank of the triangles.</td>
</tr>
<tr>
<td>20</td>
<td>1m 20s</td>
<td>Earl works with Didi and Gina as they are trying to justify Gina's claim that one triangle is twice as big as another one by using the smaller to tile the larger.</td>
</tr>
<tr>
<td>21</td>
<td>3m 40s</td>
<td>Earl comes to Mina and Tara, who are comparing triangles of equal bases and have the problem of comparing heights that are outside a triangle.</td>
</tr>
<tr>
<td>22</td>
<td>40s</td>
<td>Earl comes to Didi and Gina's table. Didi has made another claim based on measurements. He suggests that they verify it using the calculator.</td>
</tr>
<tr>
<td>23</td>
<td>1m 30s</td>
<td>Earl visits Mina and Tara, who are comparing two triangles, one of which has a larger base and the other of which has larger height. Earl suggests they measure their dimensions and calculate their areas.</td>
</tr>
<tr>
<td>24</td>
<td>11m</td>
<td>Students work independently while Earl looks at his papers at his desk. He asks groups whether they have a ranking. He turns on overhead and announces intention to discuss rankings.</td>
</tr>
</tbody>
</table>

(continued)
Summary of Lesson Segments (Continued)

<table>
<thead>
<tr>
<th>Segment</th>
<th>Duration</th>
<th>Succinct description</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>9m</td>
<td>Earl calls attention to the board and asks students to call comparisons, then asks for reasons for comparisons called. He fills an overhead worksheet using the data that students provide.</td>
</tr>
<tr>
<td>26</td>
<td>1m 10s</td>
<td>While discussing a comparison at the board, Earl brings in the actual two triangles to the projector and compares bases and heights. Then he invites Didi to comment on her conjecture.</td>
</tr>
<tr>
<td>27</td>
<td>2m 40s</td>
<td>Didi voices her conjecture and Earl illustrates it with numbers at the board. He asks Mina and Tara what they have done with $D$ and $E$.</td>
</tr>
<tr>
<td>28</td>
<td>1m</td>
<td>Earl resumes filling the worksheet with comparisons and reasons voiced by students.</td>
</tr>
<tr>
<td>29</td>
<td>4m</td>
<td>At the board Earl asks students to provide a ranking of the eight triangles. He then finds a circularity that he relates to the result of the comparison between triangles $D$ and $E$.</td>
</tr>
<tr>
<td>30</td>
<td>2m</td>
<td>Earl calls attention again to Didi’s claim hinting at the fact that these triangles might have been equal in area. He indicates that they will take on this comparison again the next day.</td>
</tr>
<tr>
<td>31</td>
<td>1m 30s</td>
<td>In preparation for homework, Earl helps students at their tables gather one set of triangles for each pair to take home.</td>
</tr>
<tr>
<td>32</td>
<td>1m</td>
<td>Earl goes back to his desk. From there he discusses the four homework problems with the class and makes suggestions on how they should approach them. Class ends.</td>
</tr>
</tbody>
</table>

Note. $m = \text{minutes. } s = \text{seconds.}$

Notes

The author acknowledges valuable comments from Deborah Loewenberg Ball to several versions of this article. He also acknowledges helpful feedback from Gary Fenstermacher and Vilma Mesa to an earlier version.

1 From the perspective of an observer, the new idea is that the formula for the area of a triangle can be used to express and calculate relationships between two unknown areas. This idea is embodied in “Didi’s conjecture” that if two triangles are such that the base of one doubles the base of the other one, but the height of the former is half the height of the latter, then their areas should be equal.

2 The triangle area formula indicates that the area of a triangle equals one half the product between the base and the height of the triangle. Any side of a triangle can be taken as the base; the corresponding height would thus be the segment perpendicular to that base through the vertex opposite the base.

3 The toolbox contained pencils, scissors, compasses, plastic right triangles, calculators, and rulers.

4 The idea of cutting and rearranging without overlap is hereafter called “break-and-make” (Freudenthal, 1983).

5 The theory does not argue for or against either of the two notions but rather acknowledges both as complementary descriptions of conditions on the work of the teacher.

6 The notion of a contract has been used to describe and explain social interactions since Jean-Jacques Rousseau (in the 18th century) wrote about the social contract (see
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Rousseau, 1994). The adjective didactical in didactical contract points to a contract that regulates those aspects of the interactions between teacher and student that have to do with the content of instruction (in contrast with interpersonal interactions).

I use at times mental state language to report on my interpretations of Earl's actions. But this choice of language does not result from an intention to uncover Earl's true motives and desires; it is rather a default choice derived from the field's lack of descriptive language for communicating about the possible meanings of human action in context (Lampert, 2001, p. 28). The purpose of the article is not to report on what Earl was truly going through as he acted (a phenomenon whose true causes might be irretrievable after the fact). Rather, the purpose is to show how the tensions proposed help provide plausible explanations for some of Earl's actions. Thus, it should be clear to the reader that my occasional use of mental state language is merely a façon de parler.

I am using the word "obligation" as a descriptive rather than normative term—not to mean normatively that a teacher should feel obliged to do so, but rather to say that everything happens as if the teacher were under such-and-such obligation.

The length of a side or base of the triangle is substituted in place of \( b \), and the length of its height (or corresponding distance to the vertex that opposes that base) is substituted in place of \( h \).

A figure is smaller than another one if it can be placed in the interior of that figure, leaving a leftover.

If a figure is smaller than a second and that one smaller than a third, the first is smaller than the third.

This anticipation might look like \( A(D) = \frac{1}{2} \cdot b(D) \cdot [2 \cdot b(E)] = b(D) \cdot b(E) \) and \( A(E) = \frac{1}{2} \cdot [2 \cdot b(D)] \cdot b(E) = b(D) \cdot b(E) \); hence, \( A(D) = A(E) \), where \( A \) is "area of," \( b \) is "base of," and \( h \) is "height of."

Quantities are things that have numberlike properties without being numbers (e.g., segments, angles). See Freudenthal (1983, chap. 1) and Thompson (1993).

It is crucial to maintain sight that there are three interrelated, substantive pieces of mathematics at play in the emergence of Didi's conjecture. Didi makes a factual claim that the triangles \( D \) and \( E \) are equal in area. She gives a warrant for that claim by making the mathematical conjecture that asserts that such a claim can be made on the conditions that bases and heights stand in the ratios 1:2 and 2:1 (see Footnote 1). And she provides backing for that warrant by noting that if those ratios were plugged into an expression of the area formula, they would cancel each other out—this backing includes the emergent idea that the area formula can be used to represent unknown quantities (the notions of claim, warrant, and backing come from Stephen Toulmin's argumentation theory; see Toulmin, 1958, and also Forman et al., 1998; Krummheuer, 1995, 1998).

The warrant is mathematically flawed in general, though the claim may be factually true for these triangles.

To be able to compare the areas of triangles whose sides have different inclinations, one needs to standardize those inclinations; thus, we customarily take the height perpendicular to the base.

One such representation could have been to consider the triangles to be compared as ordered pairs of quantities (base, height), tied by relationships such as double and half.

And this commitment was true of Gina in particular, whom Earl reported to have been happy to see very engaged with the ranking triangles task in contrast with previous lessons. Unlike Didi, Gina had not previously come across to Earl as being especially interested or talented in mathematics.

References


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