Engaging Students in Proving: A Double Bind on the Teacher

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This article uses a classroom episode in which a teacher and her students undertake a task of proving a proposition about angles as a context for analyzing what is involved in the teacher’s work of engaging students in producing a proof. The analysis invokes theoretical notions of didactical contract and double bind to uncover and explain conflicting demands that the practice of assigning two-column proofs imposes on high school teachers. Two aspects of the work of teaching—what teachers do to create a task in which students can produce a proof and what teachers do to get students to prove a proposition—are the focus of the analysis of the episode. This analysis supports the argument that the traditional custom of engaging students in doing formal, two-column proofs places contradictory demands on the teacher regarding how the ideas for a proof will be developed. Recognizing these contradictory demands clarifies why the teacher in the analyzed episode ends up suggesting the key ideas for the proof. The analysis, coupled with current recommendations about the role of proof in school mathematics, suggests that it is advantageous for teachers to avoid treating proof only as a formal process.

Key words: Classroom interaction; Communication; Geometry; Knowledge; Proof; Teaching practice

Proof occupies a central place in mathematics as a tool by which mathematicians produce, validate, and organize knowledge (Lakatos, 1976; Pólya, 1945/1957, p. 215). Proof has also occupied an important place in school mathematics. For more than a century, most geometry curricula in the United States have included opportunities for students to understand and do proofs, on the assumption that such experiences develop their analytic abilities and let them experience rigorous mathematical thinking (Herbst, in press; Moise, 1975; Usiskin, 1980).

Principles and Standards for School Mathematics (NCTM, 2000) uses the term mathematical proof to mean “arguments consisting of logically rigorous deductions of conclusions from hypotheses” (p. 56). But whereas justifying truths is one important purpose of proof, the role of proving in the discipline is more fairly described in broader terms as the internal mechanism by which mathematical concepts and properties are shaped—in particular, as the search for the conditions under which...

The episode discussed in this article belongs to a larger corpus of data that was the basis of the author’s doctoral dissertation (Herbst, 1998), completed at the University of Georgia under the direction of Jeremy Kilpatrick, who made valuable comments on an earlier draft of the article, as well. The author is also grateful for responses to earlier drafts from Humberto Alagia, Nicolas Balacheff, and anonymous reviewers.
certain assertions are true (Lakatos, 1976). Indeed, *Principles and Standards* emphasizes the importance of reasoning and proof in students’ learning of mathematics on the grounds that proofs are “essential to understanding mathematics” (p. 56).

Because of the central role of proof in the creation of mathematical ideas, *Principles and Standards* contains a recommendation that “by the end of secondary school, students should be able to understand and produce mathematical proofs” (p. 56). Yet, at the same time, *Principles and Standards* also discourages the notion that instructional units should be focused on teaching and learning proof for that sole purpose:

> Reasoning and proof are not special activities reserved for special times or special topics in the curriculum but should be a natural, ongoing part of classroom discussions, no matter what topic is being studied. (NCTM, 2000, p. 342)

As with any of the recommendations issued in *Principles and Standards*, the imperative that students should experience proving across the curriculum sets expectations not just for what students should do but rather for what students and teachers should do with the subject matter together. To learn what may be required of teachers for them to meet those expectations, it is useful to examine the place that proof has traditionally occupied in the study of geometry. The custom of “‘doing proofs’ in geometry” (NCTM, 2000, p. 56) has involved the writing of proofs in two columns of statements and reasons (see NCTM, 1989, p. 127). This two-column format has been a very visible element of the division of labor and resources between teacher and students within which they have undertaken, respectively, the teaching and learning of proof—understood as the logical deduction of conclusions from known premises (Sekiguchi, 1991).

The vision of *Principles and Standards* is presented against the backdrop of those existing practices whereby students produce proofs to practice a generic, formal skill. In recommending that experiences with reasoning and proof accompany the development of mathematical ideas throughout the curriculum, *Principles and Standards* underscores the important role that proof plays in the construction of mathematical knowledge: “Part of the beauty of mathematics is that when interesting things happen, it is usually for good reason. Students should understand this.” (NCTM, 2000, p. 56). This crucial role for proof in mathematics—explaining why interesting mathematical things are reasonable—justifies making it “a consistent part of students’ mathematical experience” (NCTM, 2000, p. 56), connected with developing mathematical knowledge, rather than an object of study by itself. Scholars interested in historical and philosophical aspects of mathematics (e.g., Hanna, 1983; Lakatos, 1976; Rav, 1999; Thurston, 1994) have produced sound arguments showing that identifying proof with a logical sequence of steps composed of statements and reasons is reductive. Studies of how students prove have demonstrated the importance, from the perspective of students’ learning, of maintaining the connections between proving and knowing (Balacheff, 1987, 1990, 1991; Chazan, 1993; Senk, 1989). Such research has questioned a formalistic representation of mathematical proof as an isolated object of study (Harel & Sowder, 1998; Moore, 1994).
However, as we think about what it means and what it takes to implement the vision of *Principles and Standards*, I suggest that there is a reason for examining traditional practice from a perspective centered on the work of the teacher. For practitioners to be willing to take on the goal of having students learn to reason and prove and to be willing to move beyond the traditional two-column format, we should be able to show the practical shortcomings of such a custom for teachers, as well as the principled objections from reformers and the apparent difficulties for students. The main purpose of this article is to provide such an argument that contributes to the development of a theory to understand the work of teaching. I examine what is expected from the teacher in the custom of two-column proving and use records of teaching practice to illustrate that work. I seek to show the plausibility of the following claim:

In two-column proving, teachers organize students’ work in such a way as to facilitate their access to the ideas of the proof. But whereas doing so may seem instrumental to enabling teachers to observe whether or not the students can do proofs, handling students’ difficulties as they uncover those ideas can become a challenge to teachers. Thus, a task that has been prepared to give students an opportunity to produce a proof can become an obstacle to a teacher who is trying to help students prove the given proposition.

My argument proceeds at two levels. At one level, I expose some of the difficulties that teachers may encounter while helping students coordinate the formal and substantive aspects of producing a proof. At another level, I demonstrate that such difficulties are necessary consequences of the specific division of labor between teacher and students that the two-column proof format imposes. The argument is a theoretical one supported by my analysis of a classroom episode in which a teacher and her students first undertake a proof of a claim about angles and then the teacher takes the production of the proof away from her students. I provide a plausible (though not necessarily causal) explanation of why such an incident can be seen as a consequence of the norms of two-column proving. Finally, I move from the detailed analysis of the episode to thinking about the work of the teacher engaging students in proving in more general terms that permit us to see similar possibilities in other cases. But first, in the following section, I formulate the theoretical constructs that help present the argument within a specific approach to the study of mathematics teaching.

**THEORETICAL FRAMEWORK:**
**UNDERSTANDING THE WORK OF TEACHING**

Explaining the teaching of mathematics is a complex endeavor. A mainstream approach searches for explanations through the empirical study of individual practitioners to describe what those individuals bring with them when they are engaged in teaching mathematics to children (Fennema & Franke, 1992; Thompson, 1992). An alternative approach to explaining the teaching of mathematics starts from an examination of the practice in which mathematics teachers are immersed, to exhibit and explain its system of relationships with mathematics and with students (Ball
& Bass, 2000; Brousseau, 1990, 1997; Chevallard, 1985; Cohen & Ball, 1999; Margolinias, 1995; Simon, 1995). In this second approach, the teaching of mathematics is often explained in terms of problems, tensions, and dilemmas that underlie the work of mathematics teachers (Ball, 1993; Cohen, 1990; Lampert, 1985). Encompassed in the work of mathematics teaching are the actions that actual teachers take in their role as mathematics teachers as well as the characteristics of a role that is structured by various contextual and institutional requirements (e.g., from the school, the profession).

The analysis that I present is inscribed in this second approach to the study of mathematics teaching and contributes to a theoretical development within a growing body of scholarship dedicated to understanding the work that teachers do in managing the production and use of mathematical reasoning and proof in the classroom (Arsac, Balacheff, & Mante, 1992; Chazan & Ball, 1999; Margolinias, 1993; Sekiguchi, 1991; Voigt, 1985; Wood, 1999; Yackel & Cobb, 1996). My argument requires two preliminary elements of theory—the notion of the didactical contract and the notion of the double bind. In the following two sections, I introduce the didactical contract and use that notion to examine the case of two-column proving, and then I introduce the notion of the double bind. The argument brings those elements together in a theoretical analysis of the work of the teacher engaging students in proving.

The Didactical Contract and the Teacher-Student Relationship

The most basic element of theory on which I build my argument is a characterization of the relationship between teacher and student within the institution of the school and in connection with the mathematics to be studied. Guy Brousseau (1984, 1986/1997, 1990) has described this relationship by saying that everything happens as if there is a contract between mathematics teacher and mathematics student—the didactical contract.1 The existence of such a contract requires the articulation of the institutional goals that link the teacher to the student and shape the subject-matter into viable activities that they undertake together to fulfill those goals. Reciprocally, the contract requires relating the actual work of teacher and students to the institutional goals that enable that work to unfold. The didactical contract thus articulates norms that are global (addressing teachers’ and students’ roles vis-à-vis the objects of study) with norms that are local (addressing teachers’ and students’ roles vis-à-vis the work they do in a given task; Chevallard, 1988). The postulate that such a contract is tacitly in place helps us understand the possible tensions between (a) the responsibility of the teacher to teach mathematics to students and the responsibility of students to learn mathematics from the teacher, and (b) the fact that in spite of their goodwill or good behavior toward one another, students and

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1 The notion of a contract has been used to analyze forms of human interaction since the 18th century, when Jean-Jacques Rousseau wrote about the social contract (see Rousseau, 1762/1994). The adjective didactical in didactical contract points to those aspects of the interactions between teacher and student that have to do with the subject matter under study.
Engaging Students in Proving

Teachers may not always be able to fulfill those responsibilities. Indeed, to make sure that students learn the requisite knowledge, the teacher needs to give them tasks on which their successful performance will indicate their acquisition of that knowledge. The teacher is thus bound to obtain (as a result of the students’ learning) rather than proclaim (as an intention of his or her teaching) a certain degree of congruence between the knowledge at stake in the tasks proposed to students and the meaning that comes from the activities in which the students engage. As one considers proof as an object of study, the notion of the didactical contract is useful in analyzing the traditional way in which the practice of teaching has made room for the imperative that students will produce proofs.

The Didactical Contract and Two-Column Proving

A brief review of the historical emergence of proof as an object of teaching and learning helps us understand the relevance of the didactical contract. Near the end of the 19th century, a curriculum reform movement made high school geometry accountable for students’ learning the “art” of proving. Several elements contributed to the assimilation of the notion of proof into the practices of teaching and learning geometry. First, a general notion of proof found its way into the curriculum as texts started to include general definitions or descriptions of proof. These descriptions emphasized the formal aspects of proof—particularly the notion that a proof is a sequence of justified statements that connect certain premises with a conclusion. The two-column format emerged later as a way of enforcing the requirement that every line of argument be justified (see Beman & Smith, 1899, p. 19; Henderson, Pingry, & Robinson, 1962, pp. 123–138; Jurgensen, Brown, & King, 1980, pp. 27–31; Schultze & Sevenoak, 1913, p. 19; Shibli, 1932; Welchons & Krickenberger, 1956, pp. 29–61). With the aid of the two-column format, textbooks used the exposition of theorems as a way of exemplifying proofs (in addition to developing the subject matter). A new kind of task came to exist, as well—the proof-exercise. These tasks were assigned to students, and their productions were expected to conform to the same formal description as the proofs in the text. Special resources were developed that facilitated the work of the teachers assigning these exercises and the work of the students doing them (e.g., conventions for giving diagrams, specialized notation, new postulates, etc.).

The conception of proof as a formal process, though reductionist, was a move that conveniently accomplished two things. At the same time that it provided a way to show similarities between the work of the teacher who was proving theorems and the work of the student who was doing proof-exercises, it permitted their work to be differentiated according to instructional needs (Herbst, in press). Thus, the propositions that students would prove could be chosen on the grounds of their being doable or short (Smith, 1911, p.70), instead of their adding important or interesting

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2 The historical details are available in Herbst (in press). I limit myself here to the conclusions that are useful to the topics in this article.
knowledge of the subject matter. “Proof” could continue to be associated with those substantive arguments used to explain the meaning of, convince somebody about, justify the truth of, or help discover a proposition that added to the knowledge of geometry. Yet, at the same time, proof would officially be any logical, deductive chain of statements and reasons derived from “given” premises and arriving at conclusions identified as “to prove.” Thus, defining proof as a formal process, embodied in the two-column format, permitted the use of the course of studies as a vehicle for teaching proof and tailoring student-centered opportunities to learn it (Young, 1906, p. 259).

Those observations are still a relevant baseline description of how proof has been taught and learned in the last few decades (Chazan, 2000; Schoenfeld, 1987; Sekiguchi, 1991). Teachers are responsible for students’ learning to carry logical deductions from premises to conclusions. The didactical contract holds teachers accountable for providing opportunities for students to practice and demonstrate that formal skill. What does this global responsibility imply for the specific work that a teacher does to have students produce a proof? I have made the claim that the division of labor entailed in two-column proving can subject the teacher to a difficult tension between contradictory demands. The argument in support of that claim draws on the notion of the double bind, which I introduce next.

The Double Bind and the Role of the Teacher

The concept of double bind comes from communication theory and was originally developed by Gregory Bateson to explain some observations of mother-child interactions (see Bateson, Jackson, Haley, & Weakland, 1956/1999; Bateson, 1969/1999). Bateson defines a double bind as a condition imposed on one of two persons by contradictory requests made by the other in a situation from which the first (called the victim) cannot withdraw. The idea of a double bind has been used in various branches of psychology and beyond—not necessarily referring to two-person relationships but, more generally, to situations involving communication between components of a system (Bateson, 1991; Rieber & Vetter, 1995; Sluzki & Ransom, 1976).

Stieg Mellin-Olsen (1987, 1991) brought the notion of double bind to mathematics education and used it to suggest, among other things, that the didactical contract puts students in a double bind. Iben Maj Christiansen (1997) examined that suggestion in a study of the simultaneous negotiation of mathematical meaning and academic tasks in a mathematical modeling course. Christiansen suggested that the students in the study had come under a double bind in which the teacher would commend them, on the one hand, for using a nonlinear model as the best fit for the real-world data they had been given, but would dismiss the model that they had produced, on the other hand, for not representing the linear real-world phenomenon from which the data presumably came. Christiansen’s work has shown the usefulness of the double bind concept in examining the students’ situation as they work on mathematical tasks that have an instructional purpose.
In this article, I use the idea of double bind in examining the work of a teacher in response to the instructional responsibilities entailed by the didactical contract related to two-column proving. In particular, I contrast the work involved in organizing a task for students to do a proof and the work involved in managing students’ production of the proof for the proposition in the given task. I propose that the didactical contract related to two-column proving imposes conflicting demands on a teacher about what he or she should do to facilitate students’ endeavors on a task that is, at the same time, an instance of doing proofs and a specific proposition demanding an argument. In the following section, I introduce a classroom episode that I use to think about this issue and to exemplify some theoretical suggestions. After the analysis of this episode, I return to the notions of the didactical contract and the double bind to revisit the argument in general.

THE RIGHT ANGLE EPISODE

The episode presented in this article is one among many from several whole-lesson video recordings that I made of a teacher whom I call Andie (a pseudonym), who was teaching an advanced ninth-grade integrated mathematics course. The unit had the dual purpose of teaching students about proof and having them study and prove some properties of angles and parallelism. My role was that of a detached observer. I did not participate in the decisions on what was to be taught. Andie had consented to be videotaped and knew of my interest in examining naturalistic classroom work to learn more about of the development of shared knowledge in the mathematics class. But she was not aware that proof, as an object of teaching, was of special interest to me. The video records of several lessons as well as the textbook unit being used are my sources for describing the context.

Andie introduced proof in two ways. When starting the unit, she referred to proof as the name for a mathematical explanation of why some things are true. The unit introduced proof as a chain of steps composed of statements and reasons leading from a premise to a conclusion, and the material offered an analogy of a person traveling from one city to another by combining and sorting available flights (Rubinstein et al., 1995, p. 394). Later in the unit, many properties of equality were identified as postulates and used to rewrite the solution of simple equations by adding justifications to each line. The rewritten solutions were called proofs. Then came the study of angles and parallelism, in which students were assigned tasks to prove many properties. When students encountered the task that I analyze here, they had already completed several proving tasks with geometric content. Whereas the appearance of acceptable proofs varied (students might write proofs in flow charts, paragraphs, or two columns), these were all graphical alterations of the two-column format, which had been introduced first. The notion that a proof was a chain of statements that started from given premises and went from one statement to the next one by citing a previously studied reason was normative.

The Right Angle Episode relies on records of a classroom discussion of a proof exercise and its solution. These records provided the raw material for a process of
inquiry into the work of the teacher engaging students in proving. This process included more than an analysis of the recorded dialogue; it involved using the recorded dialogue as traces of work that consisted of both visible and invisible features and then making hypotheses that helped delineate the invisible features. The analysis uses the elements of theory introduced earlier to explain the episode and refine the theory in that process. In the next section, I provide a descriptive account of the episode. I then use that account to formulate two questions about the demands that the didactical contract imposes on the teacher in relation to engaging students in proving the proposition at hand. In subsequent sections, I examine the episode to address those questions. Finally, I show how the notion of the double bind helps illuminate the various demands that the didactical contract imposes on the teacher.

A Descriptive Account of the Right Angle Episode

The Right Angle Episode consists of a classroom’s collective solution of an exercise that had been assigned as homework the previous day. Many students had not been able to do the exercise at home, and others had looked in the back of the book for the answer. Andie decided that the class as a whole would go over it as a way to “figure out a strategy” instead of going through the proof “without really understanding.” The exercise, taken from Rubinstein et al. (1995, p. 413), directs students to consider the diagram shown in Figure 1 and asks them to—

Write a proof of the following:

*Given:* $\angle ABC$ is a straight angle. $m\angle ABF = m\angle FBE; m\angle EBD = m\angle DBC$.

*Prove:* $\angle FBD$ is a right angle.

Andie sketched the diagram and then prompted students to produce the proof by asking them, “What do we know?” Having elicited from them that $ABC$ is a straight angle and having gotten them to conclude “without too much effort” that

![Figure 1. Diagram given to prove that $\angle FBD$ is a right angle.](image)

its measure was 180 degrees because of the “straight angle postulate,” Andie asked, “What else do we know? Cause we’ve gotten kind of as far as we can go with that.” Students then restated the two given conditions on the congruence of the two pairs of angles and justified those statements as “givens.” At that point, it seemed that nobody had anything further to say.

Andie then asked, “What are we trying to prove?”—a question that students could answer but then were unable to offer further ideas. So Andie asked, “How are we going to get to $FBD$? Do we know anything about $FBD$?” A student’s response, “Maybe it’s a right angle,” led Andie to ask whether the students knew “what makes up $FBD$.” The answer, “$FBE$ plus $EBD$,” justified by the “whole and parts postulate,” started a small discussion as to whether they were then entitled to say “$ABF$ plus $EBD$ equals ninety degrees” or whether they “[didn’t] know it” yet.

Several students took turns stating again the given conditions (transforming the phrase “equal in measurement” into “have the same measurement”) or bringing up other angles (e.g., $EBA$). Andie regained control of the discussion by saying that “the problem is, we’ve got to set something up here.” A student called Jordi (a pseudonym) then suggested that “the four angles, if you add them up, together they equal a hundred and eighty.” A new set of exchanges ensued, expressing Jordi’s idea as an equality between angles justified by the whole-and-parts postulate. Those exchanges were followed by students’ proposals of other combinations of angles using the letters in the figure. Andie wrote some of them on the board, as well as the reasons claimed. But again, in spite of the accumulation of statements from the diagram, no progress was being made toward completing the proof.

Andie took control again, erasing what she had just written and coming back to Jordi’s idea: “What’s really nice about this here is [that] we know something about angle $ABC$, that we can finally get some numbers going here…. So I’m gonna do it.” A student suggested to Andie that she write “two $FBE$,” so Andie offered the equality $2\angle FBE + 2\angle EBD = 180^\circ$. “We’re almost there now,” Andie continued. “What can we do now?” This prompted several students to suggest that they divide by 2, and a student then pointed out that to finish they needed to regroup angles $FBE$ and $EBD$ back into angle $FBD$. Andie then acknowledged that “something like this is long” and that she was not “gonna put more than two of these in the test … because they take a long time to think about.” She suggested that if they thought long enough about it—“fifteen minutes … five minutes”—they should nevertheless be able to get it “without looking in the back of the book.”

So, whereas the teacher had originally considered that the task offered a reasonable opportunity for students to produce a proof at home, it had presented serious difficulties for them. Andie acknowledged those difficulties but still upheld the value of the exercise by making it a class activity. As the class started working on the exercise, Andie played the role of a facilitator, expecting the proof to be generated by the students as she called on them for statements and reasons.

However, after the students had supplied a few of those statements and reasons, the process slowed down. The key ideas for the proof were not forthcoming. Rather, students kept voicing statements that either jumped to the conclusion
without warrant or made no progress toward it. Andie ended up suggesting some of the essential ideas for the proof. As she did this, she recast the activity as an opportunity for students to learn about proving, and she qualified the problem by identifying it as one that fell outside the mainstream. The following section uses this outline to ask two questions about the work of the teacher who engages students in proving, and an analysis of those questions revisits the data on a deeper level.

Two Questions Prompted by the Right Angle Episode

I would like to probe the episode that I have just recounted as a way of addressing more general questions about the work of teachers engaging students in proving. These questions have to do with responsibilities that the didactical contract places on the teacher. One global demand of the contract is that teachers give students tasks that enable the teachers to attest that their students can do proofs. The exercise examined in the Right Angle Episode is an example of one way in which a teacher responds to such a demand. My first question is designed to uncover what about this particular exercise recommended it to the teacher as a feasible opportunity for certifying that her students could do proofs. Thus, my first question is this: “In what way does the work involved in crafting the given task respond to the demand that the didactical contract places on the teacher to assign and evaluate tasks that, if performed successfully, will attest to the fact that the students can do proofs?”

My second question is motivated by the observation that as the episode unfolded, Andie’s role changed from a more managerial one, in which she elicited statements and reasons from her students, to a more substantial one, in which she suggested to them the main ideas for the proof. This change in the role of the teacher parallels a change in the status of the activity, which was initially presented by the teacher as a standard exercise that students were presumed to be able to do, but which ended up being regarded by all as an especially difficult activity in need of special legitimation. It is relevant to ask whether and why Andie’s suggestions were needed for the activity to proceed to completion and how they induced a change in the status of the task. Accordingly, my second question is this: “To what extent is intervening with a crucial suggestion (one that might even change the nature of the task being done) a way for a teacher to respond to the demands of the didactical contract?”

Answers to these two questions are useful for explaining more than the Right Angle Episode, which is important only by virtue of being, in many respects, so ordinary that it can be considered a typical case of a teacher engaging students in proving. I use the answers to these two questions as a means of examining what the didactical contract asks of a teacher. In that larger context, the answers demonstrate that the didactical contract puts the teacher in a double bind, a circumstance that is illustrated in the Right Angle Episode. The following two sections address each of those questions separately.
The Work Involved in the Setup of a Task

Setting up a task in a particular way is part of the work of teaching. Even if the teacher does not originate the task, the choice to use it is the teacher’s, and thus he or she is implicated in its setup. Setting up a task is work that takes place outside the classroom and is instrumental to enabling the teacher to fulfill the demands of the didactical contract. Textbook authors created the task used in the episode, but Andie gave it a place in her course. By making her students responsible for it, Andie deemed the exercise a viable way for her to respond to her obligation to give students opportunities to do proofs. The question of who created the exercise is thus, for my purposes, immaterial; what is important is that a teacher—in this case, Andie—decided that a particular exercise, crafted in a particular way, seemed to help her comply with the demands of the didactical contract. A possible rationale behind that setup—why such an exercise struck the teacher as a feasible opportunity for having students do a proof; what may have been involved in the process of making it feasible—needs to be understood to make sense of what the teacher does to have students do the task.

To uncover a possible rationale for the given task, I stepped back from the episode and identified the variables that might have been manipulated in order to create a task that responded to the demands of the didactical contract. To produce this reconstruction, I inspected the task as follows. First, I asked, “What is a question that would be answered by proving the proposition given in the exercise?” As the exercise belongs in a section of the curriculum on angles, I limited my search to a question on that topic. I called the question the fundamental question behind the given task. The question might not be unique, but choosing one that was sufficiently general would allow me to identify some variables that the task’s creators considered and some choices that they made in producing the given task. I framed the following fundamental question: What is the relationship between the bisectors of supplementary angles?

I then formulated several alternative tasks that would address this fundamental question. I looked for tasks that called for the same end product (i.e., a proof that bisectors of supplementary angles are perpendicular) but that established significantly different divisions of labor between whoever assigned the task and whoever had to undertake it. One example of such a task is the following: “Prove that the bisectors of supplementary angles are perpendicular.” That task is significantly different from merely asking the fundamental question insofar as it (a) indicates that a proof is expected and (b) states which proposition should be proved.

The process by which I came up with alternative tasks resulted in a list of instructional variables that were implicit in the design of the task given in the episode. For each of those variables, I listed the choices made in the given task. The most salient of those variables and choices appear in Table 1. To indicate that the choices could indeed vary, I also offer one alternative for each choice that was actually made in the creation of the given task. By doing so, I do not necessarily mean to imply that the alternatives are better or worse than the choices made.
Table 1

<table>
<thead>
<tr>
<th>Variable</th>
<th>Choices made</th>
<th>Alternative</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 What is the declared purpose of the task?</td>
<td>To produce a proof.</td>
<td>To investigate a question whose answer might be found through a proof.</td>
</tr>
<tr>
<td>2 What is to be proved?</td>
<td>A proposition that is explicitly given at the outset.</td>
<td>The truth value of a conjecture that students are asked to state.</td>
</tr>
<tr>
<td>3 How is the proposition stated?</td>
<td>In premises and conclusions that are identified and separated as <em>given</em> and <em>prove</em>.</td>
<td>In a compact form, as in “Bisectors of supplementary angles are perpendicular.”</td>
</tr>
<tr>
<td>4 What are the concepts whose definitions are invoked by the task?</td>
<td>Straight angle, right angle.</td>
<td>Supplementary angles (implicitly, straight angle), angle bisector, perpendicular lines (implicitly, right angle).</td>
</tr>
<tr>
<td>5 How is a diagram involved?</td>
<td>A diagram is given with the task.</td>
<td>A diagram is suggested.</td>
</tr>
<tr>
<td>6 How is the diagram labeled?</td>
<td>Letters are assigned in an alphabetical sequence supporting a circular reading of the diagram (A, B, C name the straight angle; D, E, F are the critical labels to name the four angles that make up for the straight angle).</td>
<td>Letters are assigned in such a way as to single out the angle of interest (e.g., BX bisects ∠ABE; BY bisects ∠EBC; prove ∠XBY is a right angle).</td>
</tr>
</tbody>
</table>

An Analysis of the Fundamental Question. An analysis of the mathematical work required to answer the fundamental question helps uncover the role played by the choices made in the given task. To answer the question “What is the relationship between the bisectors of supplementary angles?” one might sketch a diagram like that in Figure 2. One would consider two supplementary angles ∠AOC and ∠COB and their bisectors OX and OY, which respectively divide each of the angles into two halves (∠AOC and ∠XOC are each half of ∠AOC; ∠COY and ∠YOB are each half of ∠COB). Because ∠XOC and ∠COY have only ray OC in common, one could say that the measure of ∠XOY equals the sum of the measures of those two angles. As those were half of the measures of the supplementary angles that we started with (and supplementary angles add to a straight angle), their halves add to half of a straight angle (i.e., a right angle), which is the angle between OX and OY. The process of answering that question includes conceiving a proof. In the writing of that proof, the crucial element is an implicit treatment of the geometric situation as algebraic (i.e., treating angles as quantities; see Thompson, 1993). One combines the notion that a bisector creates two congruent angles with the assumption that angles are quantities. One calls on all the given angles, gets rid of some of them, and keeps only those relevant to determining the angle formed by the bisectors. Writing an equation like 2u + 2v = 180°, for example, could be one way to
make use of algebra, provided that \( u + v \) refers back to \( \angle XOY \). Of course, a more modest use of symbols could work just as well, as long as substitution between the geometric and the quantitative realms was made possible in order to call the given angles, distinguish the angles of interest, and “forget” the angles that were not of interest. Proving this proposition depends on understanding a situation and using resources strategically.

![Figure 2. Diagram showing two supplementary angles and their bisectors.](image)

In view of this analysis and of the choices made to set up the task, I now return to the Right Triangle Episode to show what those choices would presumably do. The general question that I want to answer here is “For the teacher, what purpose is served by the particular choices made by the textbook authors in the preparation of the task?” Thus, in the following section, I use the instructional variables listed in Table 1 and my analysis of the fundamental question to collect relevant observations from the records (i.e., to do an a posteriori analysis of the task in question; see Comiti et al., 1995) After that, I use those observations to answer the first question posed (“In what way did the work involved in crafting the given task respond to the demand that the didactical contract places on the teacher to assign and evaluate tasks that, if performed successfully, will attest to the fact that the students can do proofs?”)

An a Posteriori Analysis of the Task. The analysis of the choices that we hypothesize were made in the design of the task gives a new perspective on the recorded episode—a perspective grounded in the mathematical work that the students had to do. In listing those choices (see Table 1), I meant to answer the first part of the first question posed earlier—namely, “What kind of work was done behind the scenes to make the given exercise a viable one?” In this section, I list a number of observations of the episode that show how that work was directed to producing an exercise that the teacher could use to respond to the expectation that she engage
students in producing a two-column proof. I organize those observations according to the various choices listed in the table, explain briefly how each particular choice was helpful to the teacher in doing her job, and provide supporting evidence from the recorded episode. I then gather those observations of the episode in a summary statement addressing the first question.

• The declared purpose of the task is to produce a proof. There are two issues in the episode that help us make sense of why this choice was helpful to the teacher:
  1. As the students approached the task, they discovered that they were not required to investigate whether there was any claim that they needed to make but rather to provide a proof of a claim that was explicitly given. This fact could have served as a reason for Andie to engage students in doing the proof, in spite of the fact that an answer was already provided in the back of the book. We note that Andie was able to propose to her students, “Let’s see if we can figure out a strategy.”
  2. As the class was doing the task, this choice by its creators gave students a warrant for distinguishing whether the claim was true from whether its truth had been justified. For example, at a point when some students were claiming that it was true that the angle was a right angle, a student was able to say, “That’s what you are trying to prove.”

Thus, to require a proof made it more likely that a proof would be done, though it also made it more likely that students would not value proving as an instrument for answering a question.

• What is to be proved is a proposition that is explicitly given at the outset. The main service provided by this choice seems to be that it precludes the option of changing the statement to prove if the task becomes too difficult (and in public it precludes debates on what is the goal, should such debates arise). The value of such a choice is evident in the episode as seen in the fact that Andie could successfully rely on her students to know what the goal was when she asked them about it, even though before she asked, the students were apparently not using any goal to orient their work.

• The proposition is stated in premises and conclusions identified and separated as given and prove. Events in the episode suggest that the choice to state the proposition in this form may have been instrumental in getting the proof started: Students responded to Andie’s question “What do we know?” by restating the premises identified as “given.” Stating the proposition in this way also reinforced the notion that a proof consists of logical deductions from the premises to the conclusion by identifying a canonical starting point accessible to everyone.

• The concepts whose definitions are invoked by the task include only angles. Although this is true of both the choice that was actually made in creating the exercise and the proposed alternative, the choices offer different advantages. The given exercise has the benefit of involving a small number of relatively basic concepts; this in fact seems to help support the students’ proving work in the
Engaging Students in Proving

episode. Indeed, the dialogue shows that they understood that even definitional consequences, such as “because an angle is straight, it measures 180 degrees,” must be explicitly stated and justified in a proof. Involving additional concepts or those that are more complex might either lengthen the proof so much as to make it too complex for students or weaken the expectation that students must state and justify everything. The choice to state the proposition in simple terms was thus useful in increasing the students’ chances to participate while upholding the formal notion of proof entailed by the two-column format.

Of course, this choice would not particularly encourage students to connect this claim with other ideas or to wonder how interesting or useful the proposition was. Moreover, compared with the more elegant alternative “prove that if two angles are supplementary then their bisectors are perpendicular,” the given exercise, with its few, simple concepts, seems artificial (constructing a diagram for the given situation would necessarily involve translating the creation of at least one pair of equal angles into the halving of an angle). The given claim also seems useless (if one later had to decide whether two rays were perpendicular, the alternative might be easier to remember and apply); it is unclear what “purchasing power” the claim has in the study of angles.

• A diagram is given with the task and lettered in an alphabetical sequence supporting a circular reading. These choices seem helpful in several respects. Giving a diagram is a way to ensure that a diagram is used; moreover, it is a way to regulate how it is used. Andie and her students used the diagram in various ways in the episode. Andie called on it to represent some of the geometrical objects referred to in the premises. At one point, Andie added little arcs to all four given angles to indicate which are the congruent pairs, clarifying what was not readily apparent in the sketch on the board. In this way, she repaired the diagram so that it could support the statements about the congruence of the given angles. The diagram also provided information that was useful for the proof and was not given in any other way. For example, because the diagram showed that ray $BE$ and ray $BA$ are in different half planes with respect to line $BF$, students would know that adding them would be an acceptable step to represent a part of straight angle $\angle ABC$.

Why give a diagram? Those who created the exercise presumably decided to give a diagram in order to assist students in visualizing a claim that might be surprising to them—that there is an angle whose measure can be known even though it comes from four other angles whose measures are undetermined. If the choice had been to let the students draw the diagram (after reading a proposition like the one given), they might have required instruments (say, protractors) to ensure that the premises, such as that certain angles were equal, held. To prevent the students from then using the protractors to assert the conclusion might have been more difficult for the teacher. Students might see the fact that protractors were acceptable for part of the work but not for the rest as arbitrary. Perpendicularity (or the lack of it) might emerge as a fact that was correlated with an accurate diagram of the given situation. In contrast, since the task provides
the diagram, it imposes a temporal separation between the construction of the diagram and the use of the diagram by students to produce a proof. The choice by the task’s creators to give a diagram thus makes it easier for the teacher to preclude students’ use of empirical verification strategies (measuring, constructing) in lieu of their production of a proof.

Including the diagram also permits using the labels to serve some purposes (Netz, 1998). On the one hand, a labeled diagram facilitates stating the proposition clearly, since the statement can name objects by their labels rather than by their relationships (e.g., “\(\angle ABF = \angle FBE\)” and “\(\angle EBD = \angle DBC\)” instead of “there are two pairs of consecutive and congruent angles”). This may be instrumental in suggesting to students that they should use symbols in producing the proof. On the other hand, the way in which the task’s creators chose to assign letters promoted reading from the diagram the equality \(\angle ABC = \angle CBD + \angle DBE + \angle EBF + \angle FBA\). A circular arrangement of consecutive letters suggests that all of those angles might have some business together, even though they are not related in this way in the premises or in the conclusion. The labeling of the given figure (Figure 1) contrasts sharply with the labeling of the alternative (Figure 2).

**How the Set Up of the Task Helps the Teacher.** The way in which the task has been set up is aimed at helping the student as well as the teacher. First of all, it reduces the student’s workload by making him or her responsible for a smaller number of things. More importantly, the way the exercise is stated (form and content) and the way the diagram is involved (provided and labeled) are all variables that help make the substance of the expected proof available to students. In exchange, students are asked to focus on and care about other things—namely, the writing of a deductive chain of statements and reasons.

The fact that the task has been set up in a way that helps the students come up with the ideas for the proof also helps the teacher. Insofar as this exercise creates an opportunity for students to do something (picking up ideas for the argument and ordering them in a deductive fashion) that can be interpreted as doing a proof, it enables the teacher to claim that students who are in fact successful are doing proofs. Insofar as the choices made enhance students’ chances of success (e.g., by reducing the length and complexity of their work; suggesting ideas for their arguments), it also enhances the chances that the teacher can make that claim.

**Working Together to Prove the Proposition**

The use of an a priori analysis to examine the records of the episode permitted me to observe how some of the choices made were oriented to having students prove a proposition and helping them succeed in this undertaking. Yet, in spite of those choices, students had difficulties producing a proof. In this section, I account for the initial division of labor between teacher and student to produce this proof in class, and then I analyze the difficulties that this task presented to students. I later go back to the records of the episode to explore Andie’s handling of those diffi-
The Initial Division of Labor and the Didactical Contract. From the way in which the collective work started in the episode, it seemed as though the proof would be produced by a division of labor that had the following characteristics. Andie would use the notion that a proof consists of statements and reasons to organize a class session in which students would take turns, and she would ensure that their contributions spelled out all the elements. She would use such questions as “What do we know?” “Where do we go from here?” and “What do I have?” to elicit statements, and she would also call for reasons to support those statements by asking such questions as “Why can we conclude that?” or “By what?” She would also use this process to share responsibility for the production of those elements with as many students as possible. Students’ interventions would consist of two kinds. Students would propose statements for the proof (“Angle $FBE$ plus angle $EBD$ equals ninety”) or would give reasons for those statements (“given,” “whole parts [postulate],” or “substitution [postulate]”). They would start by stating the premises and would transform them and subsequent statements into new statements, eventually reaching the conclusion. This division of labor seems to have been normative in the episode, inasmuch as no special negotiations were made to establish it and it was initiated without comment. If this division of labor had endured, it would have supported an assertion that the students, acting as a collective, had developed the proof. But early in the development of the proof, it became clear that this division of labor was insufficient to enable the group to reach the goal.

Difficulties Inherent in Students’ Share of Labor. As indicated earlier, the choice by the task’s creators to supply a finished and conveniently labeled diagram illustrating the premises of the proposition was instrumental in supporting restrictive and productive regulations of the students’ interactions with the diagram. Their choice might have been intended to help limit students’ interactions with the diagram to a visual inspection of it, while affording them a chance to read from the diagram the statement that $\angle ABC = \angle CBD + \angle DBE + \angle EBF + \angle FBA$. But whereas the diagram might have been given with those intentions—and might afford that information—seeing the relevance of using it in that way and knowing when to use it were still problematic for students. From the students’ perspective, because of the assumption that they were doing a proof—that is, that they were deriving justified consequences from given premises—in incorporaring information from the diagram might feel like a breach of the global regulations about proof stipulated by the didactical contract. Yet, to make progress toward claiming the conclusion, the students would need to break with those regulations and do what made sense locally with respect to the proposition at stake.

But this would be a delicate move, requiring the students to analyze the meaning of the proposition to discover which things, among the many that are evident in the diagram but unstated as premises, were “fair” for them to use in the proof—and for what reasons and at what moments. Only such an analysis would protect students from claiming the conclusion before they had earned the right to do so,
but merely because the diagram supports its truth. Yet, the way in which the exercise has been set up hides from the students the background that they would need for those considerations of meaning—for example, why could it be relevant that there are two pairs of congruent angles if the target angle does not contain all of them? The setup of the exercise eases the students’ work by not making them responsible for examining the meanings behind the proof; however, those meanings may seem necessary for students to make sense of what they are asked to do.

In addition to being able to judge which observations from the figure they could use, the students would also need, later, to find a moment when it was strategic and warranted to turn from inspecting the diagram visually as a source of statements to concentrating instead on the target angle. The move from a statement about $\angle ABC$ to a statement about $\angle FBE$, in particular, is a delicate one for the students because it signals a change between possibly conflicting ways of interacting with the notion of angle being engaged by the proposition.

The students must make a transition twice from a language of familiarity with the diagram (whereby all four angles are different things) to a more functional language (Balacheff, 1987) that may enable them to simplify those observations (e.g., to consider two angles to be the same, counted twice). This transition thus implies finding an intelligent way to “forget” perceptual information from the diagram to gain knowledge about the proposition at stake. This transition is complicated for two reasons. First, after finding it productive to gather observations from the diagram, the students must move away from observing and describing toward operating on expressions that refer to the diagram only after the fact. The students must decide when it is relevant to cease observing the figure and start trusting the symbols, as well as what is relevant to preserve and what should be forgotten in representing the problem symbolically. Second, the students are expected to use algebraic tools (e.g., dividing both sides of an equation by a number other than zero) in a context where the customary symbols of algebra (say, an $x$ or a $y$) are not in place. Thus, the proof required in the Right Angle Episode demands a special quantitative language for handling the geometric objects at play, but this language is not automatically present in the conceptual system engaged by the given diagram and notation. Regardless of whether those moves appear useful, they involve another breach of the didactical contract from the perspective of the students. Since the task had spelled out exactly what had been expected and came with notation already included, the students would have found it difficult to decide that they needed yet more symbols—and of a kind that the given task had not led them to anticipate.

One can argue, of course, that if students were able to make those transitions, their performance would be valuable. The teacher would have reason to accept good performance of this task by students as an indication that they could do a proof. But how should a teacher understand students’ failure to produce a proof? The students’ manifest difficulties with the task in the Right Angle Episode raise the suspicion that steps taken in a task’s design to liberate students from the need to develop the ideas of the proof can actually become obstacles as students try to
do what they are asked. The exercise in the episode calls on the students to comply with the didactical contract—the proposition to prove is unambiguously given, they know what a proof is, and doing the task appears to be just a matter of applying their knowledge. But for them to succeed, they have to breach the contract at least twice, by taking information from the diagram to set up something that was not given by the premises and by translating from geometry to algebra to get rid of useless information. The need for the students to break the didactical contract in those two instances explains why the exercise is difficult. Indeed, building on what Mellin-Olsen (1991) suggested, we can say that global and local regulations stemming from the didactical contract issued contradictory demands on students. In the following section, I go back to the episode and describe how the way in which the initial division of labor was altered provides evidence that these particular difficulties were at play.

Altering the Division of Labor. As I have suggested, the normative division of labor for the production of the proof was affected by students’ difficulties doing their job. First, there was the moment when the class seemed not to find anything else to say beyond stating and restating premises and unpacking definitions. Some students took the occasion to claim the conclusion. For example, Jay pointed at the diagram (Figure 1) and said, “The thing about what Dan said is true, \( FBD \) … All right, \( FBA \) is equal to \( FBE \), whatever…. And then \( EBD \) is equal to \( DBC \)…. So, \( FBE \) plus \( EBD \) equals ninety degrees.” Jay was using some of the practices that are involved in making an argument (restating some statements and using so), but he was making no progress toward proving the claim. Jay’s contribution presented a problem for Andie because on the one hand it was apparently relevant to the situation—he was describing true properties of the figure and claiming the conclusion sought—and on the other hand, he was neither stating a necessary consequence nor making progress in the production of the proof. Andie responded with a blunt “you don’t know that,” and the dialogue came to a halt until Andie suggested that something had to be set up.

In addition, some of Andie’s interventions in the proof production went far beyond organizing the dialogue and instead seemed related to the need to cope with specific difficulties to ensure that the proof would be produced. Some of those interventions challenged the assumption that the students were going to prove the proposition merely by transforming the premises. In fact, Andie told them that it was time for them to do something else (“We’ve gotten kind of as far as we can go with that”; “We need to set something up here”). She also intervened to suggest to her students where to look for a statement (“We have to [keep straight to] our goal, proceed toward our goal…. What are we trying to prove?” “Do I know what makes up \( FBD \)?”). In other interventions, Andie actually took on some of the labor reserved for the students (“What’s really nice … is … that we can finally get some numbers going here…. So I’m gonna do it”).

Ultimately, Andie could not avoid acknowledging how essential her interventions had been (“Any questions about where I got something?”). This turn of events became problematic eventually, since after the proof had been completed,
students complained that it had been too difficult for them to come up with the right idea. At this point, Andie seemed compelled to qualify the task by saying that an exercise like it would not be likely to appear on a regular test: “Unless I’m going to make a test with just two of these, I’m not going to do that.” She also asked students who had done the exercise at home without looking at the back of the book how long this had taken them—probably to demonstrate that some had been able to do it in a reasonable amount of time.

These snapshots from the episode provide evidence of how difficult the moves that students were expected to make were for them. Those difficulties would undoubtedly justify Andie’s help, yet I am interested in discussing the kind of help that Andie provided. In the next section, I address the second question posed apropos of this episode—namely, whether Andie’s interventions were necessary for the activity to continue and to what extent they induced a change in the status of the task.

*The Teacher Needed to Have the Proof Done.* Could Andie have refrained from suggesting that something needed to be set up or from saying that numbers could be used? Obviously, she was not bound to say exactly what she said, but I submit that Andie could not have waited too long before intervening in some way. Furthermore, I would like to argue that Andie was in a situation where it may have seemed far more convenient to give away the ideas and use the task to make a point than to give minimal help in the hope that the proof would come from the class.

Why did Andie need to intervene? Students’ contributions had already started to oscillate between rephrasing the given conditions and jumping to the desired conclusion, as Jay’s contribution shows. For some of the students, the perception that nothing else could be concluded from the premises “without too much effort” might have served as an indication that they could claim the conclusion then and there. Also, the time that Andie let go by without intervening while they struggled might have influenced their evaluation of the complexity of the idea that they were expected to find. A lengthy delay could have increased the likelihood that students would see the task as being too difficult to be suitable for them to do on their own (see Arsac et al., 1992). I argue that as a result of those two considerations, Andie would thus feel pressed to do something to move the production of the proof forward.

What was available for Andie to do depended on how the task had been set up. Ideas for a proof had been put out for students to notice. Doing that had scripted the students’ performance (in that they were expected to use objects and notation that were given). My analysis of the creation of the task explains those choices as trying to facilitate the work of students by relieving them of responsibility for generating the ideas for the proof. Surely the work of the student is different from the work that one would do to investigate the fundamental question—though not necessarily easier. The work of the student would become less that of developing ideas for an argument and more that of noticing or uncovering those ideas in the cues given by the task and organizing them in a logical sequence.
Students might succeed in producing a proof under those circumstances, and as suggested earlier, such performances might be considered valuable. But there are reasons to expect that students might not succeed, and the lack of success should not necessarily be ascribed to their lack of reasoning ability for doing proofs. Indeed, the official notion of proof does not encourage noticing cues that are not entailed in the premises. The students’ lack of success should thus not necessarily be attributed to their not having the idea but perhaps to their not breaking the contract. Students’ lack of success might thus reveal to the teacher a mismatch between the local regulations governing the task and the global regulations of the didactical contract governing proof.

Therefore, there is an inherent uncertainty about why students might have difficulties producing a proof in the context of the task being analyzed. That uncertainty helps us understand Andie’s interventions. Whereas Andie initially followed a pattern of eliciting statements and reasons, maintaining that pattern was difficult in her situation because of two constraints:

1. If she explicitly told her students at the outset that they might need to do things that they didn’t expect or felt unauthorized to do, such as drawing information from the diagram or changing the notation, the meaning of the whole activity might depart too much from “producing a proof.”
2. If she relied on her students’ intuitive understanding of the problem, she would have no guarantee that her students would be impelled to state what was needed, when it was needed. And if they were unable to do so, they might get the message that the task was inappropriately difficult for them or irrelevant for their learning of proof.

For the teacher to fulfill his or her obligation under the didactical contract, he or she must refrain from giving the kinds of hints and suggestions referred to in (1). To keep the students from deeming the task inappropriate, the teacher might have to negotiate with them norms that are specific to the production of the required proof. But the amount of scripting involved in the setup of the task in the episode reduced the range of things that Andie could say. The more a setup makes what students will use available to them, the more costly it is for the teacher to negotiate those implicit norms without suggesting the answer.

The fact that Andie took the responsibility for the proof away from the students made it unlikely that the activity could count as “the students’ production of a proof.” In the Right Angle Episode, Andie’s statement about using numbers is as much “the moral of the story” as it is a justification for what she did. It has the effect of saying to the students, “This time you couldn’t make this statement, and I had to do it. Next time remember that numbers can help you.” From the observer’s perspective, the situation shifts from a joint production of a proof to a joint reflection on how the proof was produced. But this shift allowed Andie to give meaning to the time and effort spent in the task.
DOUBLE BIND: A GENERAL ARGUMENT ABOUT THE TEACHER’S WORK ENGAGING STUDENTS IN PROVING

One may wonder why Andie’s suggesting to her students how the proof should be done deserves so much attention. Isn’t offering a suggestion one of the usual ways in which teachers handle students’ difficulties? Indeed, it is not this ordinary fact that I present as important. What I propose as important is that the act of suggesting can be explained as an implication of the practice in which teacher and students are participating rather than as an indication of the students’ or the teacher’s deficiencies. I would like to use the detailed insights gained from the Right Angle Episode to address this issue in general and to make a connection between the notion of the double bind and the demands that the didactical contract imposes on the teacher in relation to two-column proving.

As noted earlier in this article, a double bind consists of two contradictory demands placed on a subject who cannot leave a situation. In the particular situation discussed here, the didactical contract imposes a double bind on the teacher. As in other cases in which the double bind concept has been applied, two contradictory demands exist because the activity belongs simultaneously to (at least) two contexts of different logical type (Bateson, 1969/1999). For the problem of engaging students in proving, those two contexts are (a) the task as an opportunity offered by the teacher for students to produce the proof for a proposition and (b) the proposition as a statement for which the teacher holds students accountable to find a proof. The simultaneity of those two contexts becomes problematic in relation to the contract that stipulates the teaching and learning of a general, formal notion of proof embodied in the two-column format. I examine the two contexts separately and show how they impose conflicting demands on the teacher.

Organizing a Fair Opportunity for Students to Produce a Proof

The didactical contract makes the teacher responsible to the discipline of mathematics as well as to the students. Insofar as a teacher is responsible to ensure that students have access to a legitimate notion of mathematical proof, he or she must create opportunities for them to do things that a mathematically educated observer would recognize as mathematical proofs. Two issues about mathematical proofs—the substantial one, that they efficiently handle a claim about mathematical objects (i.e., convince, explain), and the formal one, that they satisfy formal standards (i.e., derive, follow)—are likely to be influential as a teacher chooses or creates such opportunities. Yet, insofar as the teacher is also responsible to the students, he or she must ensure they have the means and the opportunity to do what is required of them. And because what is at stake is students’ acquisition of the formal skill of doing two-column proofs, students’ paths through proof-exercises should not have other obstacles.

Of course, there is no principle that prevents a teacher from asking interesting questions that lead students to make and prove conjectures that answer them. If such a thing happened, it would be fair for the teacher to claim that students have
indeed produced a proof. The question is, however, how the assignment of that activity would respond to the demands that the didactical contract imposes on the teacher. If students were to fail to come up with the statement of a conjecture or the ideas for the argument, the teacher would need to give some meaning to that failure. Uncertainty would arise about whether students were unable to reason logically or the teacher had failed to provide a fair task.

Thus, the need to interpret students’ productions in relation to the purposes of the task is likely to affect what tasks the teacher chooses for the students’ productions of proofs. The teacher’s choice of such tasks is mediated by questions that concern how students will come up with the substantive ideas for an argument that, when written appropriately, will prove the proposition. Conventions for giving diagrams, writing notation, and stating tasks have historically been used to facilitate students’ access to the ideas involved in a proof (Herbst, in press).

The first demand on the teacher is therefore that he or she choose a task that facilitates access to the ideas of the proof in some covert form. In that way, the teacher will have reason to say that the students themselves are the ones coming up with the proof—they discover the ideas and organize them deductively. Teachers will also have reason to say to students who don’t come up with the proof that their lack of success is not because the ideas were inconceivable.

**Proposing a Statement for Which the Student Must Find a Proof**

As a task is being designed to facilitate students’ access to the ideas involved in the proof sought, the work of the student is scripted. Traces of this script are left in the resources used (i.e., the diagram, the notation, etc.) but essential pieces of the script cannot be made available to the student. Students may know that things given are supposed to be relevant, but among the things that are not available to them are indications of *when* and *how* in the production of the proof the things given will be needed (as opposed to when they are not yet needed or are no longer needed). Thus, although students are asked to prove a proposition and at the same time are given covert access to the ideas for the proof, they are put in a situation similar to the one reported by Christiansen (1997). Students are explicitly asked to reason logically to prove a proposition as well as subtly invited to discover the script developed for them when the task was created.

Once again, no principle rules out the possibility that some students might reason logically and come up with a proof that happens to use all the things laid out in advance. But what is interesting to examine is what the teacher can do with the performance of those students who still have difficulties in coming up with the envisioned proof. Such a situation presents the teacher with a new uncertainty—did students fail to reason logically about the proposition to be proved or did they fail to interpret the traces of the expected argument given at the outset by the task? This uncertainty is exacerbated by the fact that those traces of the argument were not there naturally but had been included to respond to the didactical contract.
**A Contradictory Situation**

Thus, the didactical contract imposes two contradictory demands on the teacher. First, in response to the demands that he or she (a) organize a fair encounter for students with the official notion of proof and (b) reduce possible ambiguities in interpreting student performances, the teacher is pushed to facilitate students’ access to the ideas of the proof in some covert way. But when teachers follow the didactical contract in this respect, they create a source of difficulty for students and a source of ambiguity for themselves. As teachers facilitate students’ access to the ideas for the proof (through the use of particular diagrams, notation, and so on), students become accountable, at the same time, for reasoning logically about a proposition to prove and for successfully interpreting the instructional code of the task that they are to accomplish. Thus, the didactical contract may require the teacher to eliminate extraneous sources of ambiguity and difficulty—demanding that the teacher negotiate with the students norms that permit them to prove the proposition. But this negotiation is very difficult because it has to be reconciled with the need to finish a task that is already underway. The teacher is in a double bind because it is impossible for him or her to go back in time and start over. The teacher’s ability to negotiate with the student how to prove the proposition is compromised by his or her need to do something with the task at hand—that is, to come to terms with the fact that the obvious difficulties could have been caused by instructional decisions made to avoid other difficulties. Moreover, the relationship among student, teacher, and subject matter must endure in spite of those (felicitous or infelicitous) decisions.

Andie’s efforts to salvage the task by using it to suggest that whenever appropriate one should use numbers provides an example of how a teacher might deal with the double bind to preserve the relationship among teacher, student, and subject matter. Andie’s move amounts to forgetting the exercise as an opportunity for students’ joint production of a proof and affirming it as an opportunity to learn a strategy for future use in proving. This raises questions about whether it is possible to expect that a didactical contract built around the formal notion of proof embodied in two-column proving will produce fair chances for students to do things that can defensibly be called “proofs” and for teachers to be able to know what their performance means.

**CONCLUSION**

We may wonder why two-column proving might deserve so much attention. This takes us back to the issues about proving raised by the current reform movement. At first, it might seem unfair to teachers that, at the same time that *Principles and Standards* recommends giving more attention to proving, it also discourages the use of the two-column format that has apparently been such a useful instrument for teaching (NCTM, 2000; see also NCTM, 1989, p. 127). The argument presented in this article has suggested why it may actually be advantageous for teachers to look for other ways of engaging students in proving.
The commonsense impression is probably warranted—that “doing (two-column) proofs” is another one of those aspects of school mathematics where the “correct” result of the student’s work is scripted to such an extent that students’ “smart” actions are really just their noticing of cues. When students do not do that, the teacher is expected to show them what they failed to notice. If proving is to play in the classroom the instrumental role for knowing mathematics that it plays in the discipline, alternative ways of engaging students in proving must be found. The premise that knowing and proving are inseparable elements of mathematical experience is pervasive in the language that *Principles and Standards* uses to talk about reasoning and proof. In view of the argument made in this article, that emphasis should be seen not only as a challenge but also as an opportunity for teachers, who are encouraged to prevent the separation of form from substance in proving.

My analysis of the work of a particular teacher with two-column proving is thus useful in two ways. On the one hand, it has uncovered a number of issues for us to consider as we conceive of new customs for proving in the classroom. On the other hand, the method of analysis can itself serve as a guide in scrutinizing possible instructional alternatives for the reform-oriented classroom. The mandate to involve students in proving is likely to be met with the development of tools and norms that teachers can use to enable students to prove and to demonstrate that they are indeed proving. Whereas the need to accomplish those two aims in the most efficient way might recommend the development of an explicit didactical contract that will regulate once and for all what a proof is or what it means to prove, this article’s analysis provides a word of caution. It suggests that any sort of tools and norms that teachers can use to engage students in proving must allow room for the teacher to negotiate with the class what counts as proof in the context of the investigation of specific, substantive questions. Our analysis can also be a useful guide in inspecting other possible ways of making room for proof in the classroom. It prompts one to ask, for example, whether the appealing notions of argumentation (Krummheuer, 1998) or of the classroom as a scientific community (Legrand, 1988; see also Alibert, 1988; Balacheff, 1999) can be effective ways of making room for proof without imposing on the teacher a similar double bind.

In fact, there may exist multiple, effective ways of making room for proof in school mathematics, depending on the students’ level, the mathematical domain being studied, and other factors. Such diversity might initially seem to teachers much less efficient than having an explicit contract for the teaching and learning of proof. Yet, that reduced efficiency may be a fair price for teachers to pay in order to reserve for themselves a legitimate role in the work of engaging students in proving propositions that expand the base of public knowledge in the classroom (Ball & Bass, 2000).

The emphasis of *Principles and Standards* on strengthening the ties between proving and knowing in mathematics instruction from prekindergarten through Grade 12 is a great invitation to conceive of mathematics classrooms where children do better mathematics and do mathematics better (Begle, 1971). The invitation needs to be open to conceive of a viable role that the teacher could play in shaping mathematical activity in classrooms without taking over the students’ work.
REFERENCES


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