When is an Argument just an Argument and when is it Proof?
Mathematical Argumentation as a Window to Students’ Ways of Reasoning

Kay McClain
Vanderbilt University

DRAFT PLEASE DO NOT CIRCULATE

Dear Colleagues,

For some of you, I have taken great liberties with the term *proof*. One of my goals for the conference is to gain a better understanding of an appropriate use of this term as it relates to mathematical argumentation. So, if in your reading you will indulge me, I will look forward to discussions around this issue in September.

Kay

Support was provided by the National Science Foundation under grant no. REC-0135062
The episodes featured in this paper were also the subject of analysis in:

Abstract

This paper uses analysis of two classroom episodes to clarify the important role of mathematical argumentation as a segue to proof. The goal is to clarify the role of the teacher and the teacher’s mathematical understandings in shifting whole-class discussions toward a rigorous stance of proof. The episodes featured in this paper provide a rich setting in which to investigate not only the influence but also the confluence of students’ and teacher’s understandings on the quality of whole-class discussions. In particular, I focus on the communication between the students and myself as the teacher. This unique perspective allows me to offer insights into the teacher’s decision-making process and how those decisions influenced the opportunities for learning. The analysis will therefore make explicit the complexities in teaching by highlighting the importance of the teacher’s understanding of the students’ offered explanations and justifications and the mathematics that is to be taught in shifting discussions toward a stance of proof.
Research on effective teaching often characterizes the teacher’s classroom decision-making process as informed by the mathematical agenda, but constantly being revised and modified in action based on students’ contributions (cf. Ball, 1990; Ball, 1993; Carpenter & Fennema, 1991; Cobb, Yackel, & Wood, 1991; Lampert, 1990; Maher, 1987; Simon, 1997; Simon & Schifter, 1991; Thompson, 1992). Balancing the tensions inherent in simultaneously attending to students’ offered solutions and the mathematical agenda is the hallmark of deliberately facilitated discussions (cf. McClain, 2003). These discussions involve a plethora of decisions that must be made both prior to and while interacting with students. The image that results is that of the teacher constantly judging the nature and quality of the students’ contributions against the mathematical agenda in order to ensure that the issues under discussion offer means of supporting the students’ mathematical development. This view of mathematical discussions stands in stark contrast to open-ended sessions where all students are allowed to share their solutions without concern for potential mathematical contributions. In these latter settings, mathematical argumentation is rarely achieved.

In order to engage in the process of elevating discussions to the level of mathematical argumentation that constitutes the basis of proof, the teacher must have a deep understanding of the mathematics under discussion (cf. McClain, 2004). This is critical in both being able to advance the mathematical agenda and in judging the quality and worth of student contributions. It requires decision-making in action concerning the pace, sequence and trajectory of discussions in order to ensure that topics under discussion move the mathematical agenda forward.
The importance of a strong knowledge of content in the teaching of mathematics has been acknowledged by a variety of scholars (Ball, 1989; Ball, 1993; Ball, 1997; Bransford et al., 2000; Grossman, 1990; Grossman, Wilson, & Schulman, 1989; Ma, 1999; Morse, 2000; National Research Council, 2001; Shulman, 1986; Schifter, 1995; Sowder, et al., 1998; Stein, Baxter, & Leinhardt, 1990). In doing so they emphasize the importance of a deep understanding of the mathematics one will teach so that decisions made in action about appropriate avenues of exploration can be framed against a relatively sophisticated understanding of the mathematical concepts to be explored. Teachers therefore need to have a strong sense of what constitutes the mathematical basis of each investigation that they conduct in their classroom. The work of Carpenter and his colleagues in Cognitively Guided Instruction (Carpenter, Fennema, & Franke, 1996; Carpenter & Fennema, 1992; Fennema, Carpenter, Franke, & Carey, 1993) point to the importance of teachers understanding not only the mathematics that they teach, but also their students’ mathematics. It is only in understanding their students’ ways of reasoning that teachers can support discussions that are characteristic of argumentation and proof.

When focusing on students’ offered explanations and justifications, the teacher is seen to actively guide the mathematical development of both the classroom community and individual students (Ball, 1993; Cobb, Boufi, McClain, & Whitenack, 1997; Cobb, Wood, & Yackel, 1993; Lampert, 1990). This guiding necessarily requires a sense of knowing in action on the part of the teacher as he or she attempts to capitalize on opportunities that emerge from students' activity and explanations. With this comes the responsibility of monitoring classroom discussions, engaging in productive mathematical discourse, and providing direction and guidance as judged appropriate. Similar
pedagogical issues are addressed by Lampert (1990) in her discussion of the teacher’s role in guiding mathematical argumentation as a zigzag between conjectures and refutations. This characterization of the role of the teacher is also evident in Simon’s (1995) account of the Mathematics Teaching Cycle, which highlights the relationship between teachers’ knowledge, their goals for students, and their interaction with students. Teachers who approach teaching as a generative problem-solving activity in the course of which they modify their goals, understandings of students’ thinking and of mathematics, and the means they use to support students’ mathematical development contribute to the improvement of classroom practice (cf. Ball, 1990; Carpenter & Fennema, 1991; Cobb, Yackel, & Wood, 1991; Franke, Carpenter, Levi, & Fennema, 1998; Maher, 1987; Shulman, 1986; Simon & Schifter, 1991; Thompson, 1992). A focus on the importance of students’ contributions also highlights the importance of norms that constitute the classroom participation structure (Erickson, 1986; Hershkowitz & Schwartz, 1999; Lampert 1990; Simon & Blume, 1996; Sfard, 2000; Voigt, 1995; Yackel & Cobb, 1996). The importance attributed to classroom norms stems from the contention that students reorganize their specifically mathematical beliefs and values as they participate in and contribute to the establishment of these norms (cf. Bowers & Nickerson, 1998; Lampert, 1990; Simon & Blume, 1996; Voigt, 1995).

In the analysis in this paper, I focus on communication between the students and myself as the teacher in the classroom and how my understandings both supported and constrained that communication. The analysis will therefore make explicit the tensions in teaching by highlighting the importance of the teacher’s understanding of the students’ offered explanations and justifications and the mathematics that is to be taught.
Data

The episodes analyzed in this paper are taken from a classroom design experiment conducted in the fall semester of 1997 with a group of twenty-nine American seventh-grade students (age twelve). During the twelve weeks of the experiment, the research team assumed total responsibility for the class sessions, including myself as teacher. It is therefore important to note that my analysis is grounded in both a documentation of my decision-making process as recorded in my daily journal and a retrospective analysis of these decisions.

The primary goal for the classroom design experiment was to investigate ways to proactively support middle-school students’ ability to reason about data while developing statistical understandings related to exploratory data analysis. An integral aspect of that understanding entailed students coming to view data sets as distributions. In that process, they would structure and organize data sets multiplicatively as they created databased arguments that were grounded in their analysis.

Method

For the purposes of this paper, I will focus on the sociomathematical norms that became negotiated in the project classroom (cf. McClain & Cobb, 2001; Yackel & Cobb, 1996). Sociomathematical norms include what counts as a different mathematical solution, a sophisticated mathematical solution, an efficient mathematical solution, and an acceptable mathematical explanation and justification (cf. Herschowitz & Schwartz, 1999; Lampert, 1990; Simon & Blume, 1996; Voigt, 1995). In contrast, social norms can
be characterized as general norms that are necessary for engaging in classroom discussions and can apply to any subject matter. Such norms include explaining and justifying solutions, attempting to make sense of explanations given by others, and challenging others’ thinking (Cobb & Yackel, 1996).

In particular, analyses in this paper will focus on the evolution of the sociomathematical norm of what counts as an acceptable mathematical argument, the norms for justification for these arguments and how they provided the basis for a stance of proof. However, a focus on the sociomathematical norm of what counts as an acceptable mathematical argument makes assumptions about the establishment of certain social norms. It is therefore important to note that both the social and sociomathematical norms were a focus of attention throughout the teaching experiment.

Analysis of Classroom Episodes

Before providing analyses of the classroom episodes, I provide background information on the processes employed by the research team prior to entering the classroom and its importance to me as the teacher. In particular, in the months prior to the classroom teaching experiment the research team mapped out a hypothetical learning trajectory that guided the development of the instructional sequence. It was a conjecture about the learning route of the classroom community and was constantly subject to change and modification based on ongoing analysis of the classroom sessions.

As the research team began to design the instructional sequence to be used in the seventh-grade classroom, it attempted to identify the “big ideas” in statistics. The plan was to develop a single, coherent sequence and thus tie together the separate, loosely
related topics that typically characterize American middle-school statistics curricula. In doing so, the research team came to focus on the notion of distribution. This enabled measures of center, relative frequency as well as other features such as “skewness” and “spread-outness” to be treated as characteristics of distributions. It also allowed various conventional graphs such as histograms and box-and-whiskers plots to be viewed as different ways of structuring distributions. The instructional goal was therefore to support students’ gradual development of a single, multi-faceted notion, that of distribution, rather than a collection of topics to be taught as separate components of a curriculum unit.

Integral to the trajectory were conjectures about how to support this development. This included design decisions about tasks and the development of computer-based tools for analysis. As a result, the two tools were designed to offer resources that would support the emergence of the conjectured learning trajectory.

This conjecture about a learning route and the means of supporting it guided my decision-making process as I interacted with students in the classroom on a daily basis. For that reason, I was continually attempting to gauge the students’ current understandings against the mathematical agenda. It was therefore critical that I keep the envisioned mathematical endpoint as my guide. At the end of each class session, the research team discussed the day’s events and made necessary revisions to the instructional sequence based on observations of the students’ activity. This required that I not only have a deep understanding of the mathematical purpose of each classroom task, but that I also understand students’ solutions in order to be able to make reasoned
decisions about how best to proceed. In doing so, I placed significant importance on the role of mathematical argument as a window to students’ thinking.

**Batteries Episode**

As will become apparent from my analysis of the *Batteries* episode, I initially found it very difficult to balance these multiple agendas. As a result, I was unable to communicate effectively with the students during discussions. *My* understanding of their understanding was confounded by my interpretation of their use of the computer tool. Because of my preconceived judgments and anticipations about solutions for the task and efficient tool use, I had difficulty adjusting to theirs. I was attempting to only *influence* instead of also *be influenced by* the students’ actions and constructions (cf. Steffe & Thompson, 2000). For this reason, my preconceptions about efficient tool use constrained my ability to capitalize on the students’ activity and served to delimit the effects of the subsequent discussions.

**Classroom activity structure.** Before proceeding with the analysis, it is useful to clarify the classroom activity structure. This structure was composed of three general phases, often spanning more than one class period. The first of these involved the introduction of the task. Since the research team decided to use archival data in the teaching experiment, we judged that it would be important for the students and myself to talk through what the research team came to call the data creation process (cf. Tzou, 2000). In effect, we wanted the students to elaborate the design specifications that would yield the data they were to analyze. This aspect of the classroom was typically characterized by students clarifying for themselves the question to be answered or dilemma to be resolved and the ramifications of the data collection procedures on their
When is an Argument just an Argument?

analysis. During these discussions, the students would create a list of attributes on which they would want to generate measures in order to answer the question. This was followed by discussions of how to collect the data, what measures would be appropriate, and how to account for differences in data collection procedures. This process proved important in grounding the students’ activity in the context of a situation that had real consequences.

My initial interpretation of this phase of the classroom activity structure entailed my telling the students a story about a situation that would ultimately require the needed analysis. In these instances, the students often recounted incidents from their personal experience, but did not actually engage in the process of data creation. After these discussions, I presented the data the students were to analyze and attempted to support their understanding of the process by which it was generated by offering an explanation. As a result of their not having engaged in working through the data creation process, their analyses often lacked grounding in the context of the question at hand. This created a limited basis on which to initiate substantive discussions – be they arguments or proofs.

As an example, in introducing the batteries task, I asked the students if they were familiar with different brands of batteries. I followed by asking them if they knew what most battery companies “claimed.” After some suggestions (guesses?), one student noted that most of them claim to last the longest. Another student stated that some of them had to be lying if they all claimed to last the longest. There was an extensive discussion about this and students even talked about the “small print” that companies use when they explain how the batteries were tested. I then asked the students to make suggestions as to how we might test batteries to determine their longevity. They responded by suggesting
that the different brands of batteries be placed in similar devices (e.g. flashlights) and
timed to see how long the device would run.

Although I engaged the students in a conversation related to batteries and how we
might test them, the students did not discuss what data would be needed to make a good
decision (e.g., data on the time that the batteries lasted). I imposed those criteria. As a
result, their framing of the task was not grounded in a process of data creation that would
generate measures to be used in answering the question at hand. They only responded to
my questions about how to test for longevity. This in turn contributed to some of the
students’ subsequent inability to situate their analysis in the context of the investigation.

The second phase of the classroom activity structure involved the students
working in pairs or small groups at the computers to analyze the data. During this phase
I would circulate among the groups of students to monitor their work. My role was not to
intervene and correct students’ errors or to suggest strategies for the particular task. It
was instead to gain an understanding of the varied and diverse ways students were
approaching the task so that I could plan for the subsequent whole-class discussion. To
this end, I would identify students whose solutions, when discussed, could support the
emergence of the mathematical arguments that could be elevated to conclusions or proof.
This might entail comparing and contrasting two or more solutions or discussing a
sequence of solutions intended to build toward a mathematical endpoint.

The third and final phase of the classroom activity structure entailed a discussion
and critique of the students’ analysis. During this phase, my role was to deliberately
facilitate the discussion by selecting students to share their solutions and highlighting
aspects of those solutions that were mathematically significant. Simultaneously, I was
attending to the negotiation of norms for mathematical argumentation. This highlights the importance of the relationship between the negotiation of classroom social and sociomathematical norms and the students’ mathematical development.

Each of the three phases of the classroom activity structure was dependent upon effective communication between the students and myself. As will become apparent from my analysis of the *Batteries* episode, my understandings and interpretations of the students’ explanations and tool use often interfered with this communication. I was unable to make adjustments to my interpretations in order to accommodate the students’ ways of reasoning. For this reason, the discussions failed to provide an effective venue for productive mathematical discussions or arguments.

**Instructional intent of the batteries task.** The batteries data was presented to the students in the first of the two computer tools. The tool displayed the data values as horizontal bars, the length of which corresponded to the magnitude of the measure as shown in Figure 1.

[Insert Figure 1 here]

As this was only the second task presented using the first computer tool, a primary goal of the task included supporting students’ ability to act on the bars in the tool as data. This was to ensure that the students’ activity take on the characteristics of genuine data analysis instead of simply manipulating numbers in an attempt to complete an assignment. For this reason, situations were chosen in the initial investigations such that the attribute measured had a sense of linearity (e.g. braking distances of cars and length of time a battery lasted). It was conjectured that the bars would therefore be a linear representation of the measure.
Prior to introducing the batteries data, I worked to understand the instructional intent of the task. This was necessary to guide my decision-making process as I interacted with the students. This involved not only an understanding of how the task fit in the instructional sequence as outlined by the conjectured learning trajectory, but it also involved making anticipations about how students might reason on the task and subsequently structure the data as they worked with the computer tool.

A second aspect of this goal was that the students would find ways to characterize the data that attended to the question. This was based on analysis of pre-assessment data that indicated that the students typically calculated the mean when comparing two sets of data. The goal was to create a perturbation in this way of thinking and initiate shifts toward ways of reasoning that focused on features of the data sets including variability. Features on the computer tool that allowed the students to order and partition the data supported this goal. The goal was then that the students would come to reason about the characteristics of the data as they created arguments based on comparisons across the data sets.

The research team also had an overarching goal of supporting students’ ability to reason multiplicatively about data (cf. Harel & Confrey, 1994; Thompson, 1994; Thompson & Saldanha, 2000). Multiplicative reasoning is distinguished from reasoning additively. As an illustration, in reasoning additively, students might focus on the absolute frequency of the data in particular intervals (e.g., 15 cars are going faster than the speed limit and 30 are going slower). In contrast, in reasoning multiplicatively, students might focus on the relative frequency or proportion of the data (e.g., one-third of the cars are going faster than the speed limit). As a result of our noting the importance of
this distinction, I had anticipated that students might use the computer tool to partition
the data sets and reason about proportions of the data that fell within or above a certain
range or cut point. These would be solutions on which I would want to capitalize in the
course of whole-class discussion in order to elevate the discussions of acceptable
mathematical arguments.

Classroom analysis. As the students worked at the computers on their analyses, I
circulated among the groups to identify students whose ways of reasoning would support
the emergence of the afore mentioned goals. In doing so, I took a “snap shot” of their
activity and attempted to determine how they were proceeding with their analysis as I
made decisions about how to organize the subsequent whole-class discussion. From my
brief exchange with Carol and her partner, I inferred that they had partitioned the data set
with the range feature on the computer tool and reasoned about proportions of each brand
of battery on either side of the cut point. I judged this to be an efficient and sophisticated
way to approach this task and one that supported my mathematical goals. I therefore
selected Carol to share in hopes of highlighting this solution method in the whole-class
discussion. My judgment about their solution was based in part on the way Carol and her
partner had organized the data with the computer tool. They had used the range feature
to identify the top ten values as shown in Figure 2.

[Insert Figure 2 here]

I interpreted their use of the range feature as a way to partition the data. In doing
so, I assumed that Carol and her partner had reasoned about proportions of each data set
that fell within a certain range. However, when Carol gave her explanation, it was clear
that I had misinterpreted their activity. This could be due in part to my brief interaction
with the students, but also in part to the fact that I had interpreted their activity in a manner that fit with how I would attempt to analyze the data. As a result, Carol’s explanation was unanticipated. This was a pivotal event, placing me in the position of first ensuring that I understood the students’ solution and then re-acting to their explanation. I was therefore no longer able to pro-actively orchestrate the whole-class discussion.

While it could be argued that Carol and her partner did in fact partition the data with the range feature, the particular argument Carol offered did not fit with my interpretation of an argument based on partitioning. Carol stated that she used the range feature to identify the top ten batteries out of the twenty that were tested and noted that Always Ready were more consistent with “seven out of the top ten.” For me, this argument would involve attention to the proportion of each data set that fell within a certain range. However, the choice of the top ten was, for Carol, a valid basis for selecting Always Ready as the better battery. I, however, was further confused by her use of the term consistent while simultaneously focusing on the top ten. At this point I was wondering if I understood her correctly. For that reason, I asked Janine to explain her understanding of Carol’s solution to me.

**Janine:** I understand.

**Kay:** You understand? Ok, Janine, I’m not sure I do. So could you say it for me?

**Janine:** She’s understanding, I mean she’s saying that out of ten of the batteries that lasted the longest, seven of them are green, and that’s the most number, so the Always Ready Batteries are better, because more of those batteries lasted longest.
Janine’s interpretation of Carol’s explanation also did not fit with mine. I was now focused on Carol’s use of the term consistent. For me, this implied a smaller range. This was not, however, the explanation that Janine gave of Carol’s solution nor did it fit with the notion of consistency. At this point in the discussion, I felt that I was faced with either imposing my interpretation of how to more effectively use the range feature, or simply calling upon another student without attending to Carol’s solution method. Neither of these moves would support the emergence of mathematical arguments grounded in analysis of the data and would mimic a “show and tell” pattern of discourse.

Fortunately, Juan was able to pose a question that allowed me to explicate my concern. Juan noted that if Carol had chosen to use the range feature to capture the next “bunch of close ones” (e.g. the next four batteries) then there would have been seven of each brand represented within the range feature. At this point, I built off of Juan’s question and attempted to get Carol to offer a justification for her choice of the “top ten” instead of, say, fourteen. However, when she responded that her selection of the top ten data values was based on “trying to go with half,” I was unable to think of a question to challenge her further.

As I deliberated about how to proceed, Brad raised his hand to say that he solved the task a different way. Unable to build from the current discussion, I allowed the shift. Brad had used the value bar to partition the data at eighty hours and had reasoned about the parts of each set that fell above eighty hours as shown in Figure 3.

[Insert Figure 3 here]
In doing so, he had judged Tough Cell to be the better brand because he would “rather have a consistent battery that I know that’ll get me over 80 hours than one that just try to guess.” I followed up by asking Brad why he chose 80 hours as a cut point. He stated that his choice was based on the fact that “most of Tough Cell batteries were all over 80.” I interpreted him as viewing 80 hours as a lower limit.

In retrospectively attempting to tease out important differences between Carol’s and Brad’s explanation, I frame both in terms of Toulmin’s scheme for argumentation (see Figure 4).

[Insert Figure 4 about here]

Carol and her partner analyzed the data and reached the conclusion that the Always Ready batteries were better. When asked to give a warrant, or to explain how they reached their conclusion, Carol stated that she looked at the top ten of the twenty batteries. Juan prompted me to ask Carol for a backing, or a justification for her warrant. The backing she offered was that she “was trying to go with half.” A similar exercise with Brad’s explanation finds his conclusion different. He chose Tough Cell. Brad’s warrant or explanation for his conclusion was that “all the Tough Cell is [sic] above 80.” When asked to provide a backing or a justification for his warrant, he explained “I’d rather have a consistent battery that I know that’ll get me over 80 hours than one that just try to guess.”

A comparison of the two backings offered offers an opportunity to highlight an important shift that occurred in the course of the discussion. Although the students understood Carol’s procedure (looked at the top half of the data and noted which brand had the most results there), her choice of the top ten did not appear to be a valid statistic
or backing. In particular, Juan argued that if you modified the position of the range feature and chose the top fourteen batteries instead of the top ten, you would have seven of each brand, as the next four batteries are Tough Cell. Choosing the top ten was an arbitrary choice; not valid for the investigation at hand. Brad, on the other hand, gave a backing for choosing 80 hours that appeared to be valid for all of the students. He wanted batteries that he could be assured would last a minimum of 80 hours. In particular, his backing was grounded in the situation-specific imagery of the longevity of the batteries.

In this discussion, the students were beginning to negotiate what constituted a sufficient backing or acceptable mathematical argument in this classroom. Although there is no attempt on my part to clarify the need for a backing that is grounded in the context of the investigation, this does emerge from Juan’s challenge. Juan wanted to know what was significant about choosing ten; why not fourteen. I, However, was able to capitalize on Juan’s comment by questioning Carol about her choice of ten. In addition, the challenge by Juan also prompted me to ask Brad to justify his decision to partition the data at 80 hours. As a result of the students pushing for an adequate backing, the forms for argumentation shifted slightly to include a backing grounded in the question at hand. It was insufficient to describe a procedure. Activity had to be grounded in the question. This small shift was a first move toward argumentation that could constitute proof.

The exchange continued with Janine questioning Brad’s choice of Tough Cell as more consistent. Janine claimed that Always Ready was also consistent. In retrospect it appears that Janine and Brad were using different criteria for judging consistency. In doing so they were focusing on different features of the computer tool. In order to clarify this difference it is necessary to explain that on the previous task another student, Will,
first introduced the term consistent when using the same feature on the computer tool. Students had been asked to judge the comparative safety of two makes of cars by looking at the braking distances of ten of each make. Will had based his argument on consistency stating that it would be important to know each time you applied the brakes, about how long it would take to stop. Therefore the more consistent data set was the better make of car. He had used the range feature to identify the extreme data values in each data set in order to determine the range and then to make comparisons. In the batteries task, Carol and her partner had structured the data using the same feature on the computer tool and had reasoned that since seven out of ten Always Ready were in the top ten, it was more consistent. Although Janine’s recasting of Carol’s solution did not involve the use of the term consistent, it appears that in her conversation with Brad she was referring to the data structured as in Carol’s solution (e.g. with the range feature as shown in Figure 2). Evidence of this can be found in her reference to “only three out of ten” which is the number of Always Ready batteries falling outside of the captured range. However, Brad was arguing for Tough Cell based on the fact that all ten lasted longer than 80 hours. Brad’s response to Janine was based on his way of structuring the data with the value bar as he stated, “you still have 2 that are behind 80 in the Always Ready.” It is only in retrospect that I became aware of my contribution to the lack of communication.

At this point in the discussion, there is also an indication that Brad and Janine are talking past each other – there is a breakdown in communication. Brad and Janine have different understandings of the term consistent and are applying them to the data structured in two different ways. I contributed to the miscommunication by leaving both features of the tool visible (e.g. the range feature and the value bar) as shown in Figure 3.
This therefore constrained the students’ ability to attend to the two different ways of reasoning about the data. Brad’s interpretation of consistency was based on the number of data values greater than 80 hours as defined by the value bar. Janine’s was based on the number of data values in the top ten as captured by the range feature.

As the conversation continued, the lack of communication was further complicated by the fact that for Brad, the batteries being analyzed were a representative sample of the population. I, in fact, recast his interpretation in terms of being representative. However, Janine, who was still focused on the top ten batteries, did not understand why a similar thing would not happen with the Tough Cell batteries. For her, the data sets as inscribed on the computer tool were the population. This appears to also be the case for both Jasmine and Suzanne who spoke about getting “one of the bad batteries” or “the ones that were lowest.”

Throughout this episode, there is evidence of the break down in communication between the students and myself. Initially, I was trying to make sense of the offered solutions for myself. This occurred after Carol’s explanation and again after Brad’s. In the latter case I rephrased Brad’s decision to partition the data at 80 hours in terms of a lower limit. In this instance I was verbalizing my understanding of Brad’s method. However, I made no attempt to ensure that other students understood these explanations. This lack of attention to students’ understanding is further highlighted by analyzing the exchange between Brad and Janine that occurred at the end of Brad’s explanation.

**Janine**: Um, why wouldn’t the Always Ready batteries be consistent?

**Brad**: Well, because all your Tough Cell is above 80, but you still have 2 that are behind 80 in the Always Ready.
Janine: I know, but that’s only 3 out of 10.

Brad: No but see, they only did, what 10 batteries? So the 2 or 3 will add up. They’ll add up to more bad batteries and all that.

Kay: Oh. I see. As you get more and more batteries it’s going to get more, more bad ones if that’s representative. Ok, is that? Janine?

Janine: So why wouldn’t that happen with the Tough Cell batteries?

Brad: Well, because the way that those 10 batteries show on the chart that they’re all over 80 that means that it seems to me that they would have a better quality.

Although I was aware of the sophisticated nature of reasoning about samples, I made no attempt to ensure that either Janine or other members of the class understood Brad’s explanation.

This lack of communication is consistent throughout the first part of the episode and is confounded by the tool use. I made no attempt to clarify which way of structuring the data the students were referencing in their arguments. This is a major contributing factor to the ineffective nature of the discussions. Because both structuring features remained visible throughout the discussion, there was a continued lack of clarity in the arguments. Students only saw the aspects of the inscription that were pertinent to their argument.

In reflecting back on this episode, it is clear that initially I had interpreted the students’ contributions against my more sophisticated understandings of the tool and the task. I was giving too much agency to their activity on the computer tool. As an example, when one student ordered the value bars as shown in Figure 1, I saw the shape
of the distribution and assumed it was transparent for her also. In coming to understand
the importance of taking account of *their* actions and constructions, I began to work
much harder at making sense of their activity with the tool instead of imposing my
understandings. This supported a shift in the norms for argumentation and what counted
as an acceptable mathematical argument.

**Transition to AIDS Episode**

In the weeks following the batteries episode, as the students engaged in a
sequence of data analysis tasks, I continually reflected on my activity against the
background of the students’ learning and the overall intent of the instructional sequence.
This occurred through daily journal writings and interactions with other members of the
research team. In doing so, I came to understand the importance of deliberately
facilitated whole-class discussions and the norms for argumentation that would support
students’ active participation in these discussions. As a result of my continually
developing understandings, there was a gradual shift in my ability to get beyond my
interpretations of the task and focus on the interpretations of the students. In doing so, I
acknowledged the importance of communication between the students and myself and the
role of the tools in supporting that communication. This shift in my understanding of the
importance of the students’ ways of reasoning can be seen in the AIDS episode that
occurred twenty-three class sessions later. In this episode, I am able to build from the
students’ analyses to continue to renegotiate what counts as an acceptable mathematical
argument and move towards a stance of proof.

**AIDS Episode**
McClain, 2004

*When is an Argument just an Argument?*

The task for the students in the *AIDS* episode varied in several ways from the one in the *Batteries* episode. First of all, it was inscribed in the second of the two computer tools. This tool displayed data as line plots and contained numerous features for structuring the data. In addition, two sets could be positioned over each other for comparison as shown in Figure 5.

[Insert Figure 5 here]

Over the course of the teaching experiment, the research team decided that asking the students to choose the better brand or better treatment focused the students’ analyses on the choice. By asking the students to think of a way to structure and organize the data so that someone else could make a reasoned decision, the students were asked to investigate patterns in the data and then decide how best to represent those to someone who would not see the data sets in their entirety. For this reason, discussions focused on features of the students’ inscriptions. As students typically agreed on the “better” case, this reformulation of task provided an opportunity to shift the nature of the mathematical arguments from *which is better?* to *Which is the better argument?* In this way, students engaged in reflective discourse (cf. Cobb, Boufi, McClain, & Whitenack, 1997) about their prior activity. As part of this process, students began to focus on refining the argument in order to strengthen the case instead of simply arguing about the choice.

As an example, one option the students used to structure the data was the *create your own groups* in order to determine the actual number of data values above and below a cut point. The resulting inscription would be a line with the location of the cut point noted, the extreme values, and the number of data values appearing in each interval recorded as shown in Figure 6.
When making comparisons, students would stack these (similar to the way they were stacked on the computer tool) and reason across them in order to judge which brand or treatment was better. This judgment would be based on how the data were distributed to the two intervals. In most situations, it was not unusual for all of the students to agree on a brand or a treatment and still engage in lengthy discussions about the best way to represent the results of their analyses. For this reason, the nature of the argument changed from *which is better?* to *How is it best structured to make the argument?* This caused a substantial shift in the norms for argumentation in that the correct choice was subjugated to more mathematically significant arguments about the features of the distributions.

During the introduction of the AIDS task situation, it became evident that the students were quite knowledgeable about AIDS and understood the importance of finding an effective treatment. Further, they clarified the relation between T-cell counts and patients’ overall health (increased T-cell counts are desirable). They had studied AIDS in Health class and were aware of social issues surrounding treatments and the importance of a cure. After the discussion, I introduced screen captures of the data inscribed in the second computer tool as shown in Figure 4.

At this point in the instructional sequence, the goal was to support shifts in toward mathematical arguments about the data structured multiplicatively. This particular task was selected because the two data sets contained an unequal number of data points. In particular, one data set was much larger than the other (46 patients compared to 186 patients). In earlier tasks with unequal numbers of data points, students spent a great deal
of their time deciding which data points to ignore in order to have equal N’s. One goal of this task was to create a situation where equalizing the sets by eliminating points was not possible. It was hoped that this would create the need for multiplicative ways to structure the data in order to make a comparison since direct additive comparisons would be inadequate.

As the students worked in pairs on their analyses, I again monitored their activity in order to plan for the subsequent whole-class discussion. In this particular instance, the discussion of the data creation process and the students’ analyses encompassed the entire class period. As a result, I was able to collect the students’ reports and inscriptions and use them in making decisions for the following day. In working through their solutions, I noticed that some students found qualitative ways to make comparisons between the two data sets such as referring to where the “majority” of the data were located. Other students found quantitative ways to make the distinctions by noting the number of data values above and below a certain cut point. Still others had structured the data into four equal groups and reasoned about the percentages of each data set that fell above a certain T-cell value. This range of ways of reasoning from qualitative comparisons based on perceptual patterns in the data to the data structured multiplicatively was a perfect venue for initiating shifts in the students’ ways of reasoning toward inscriptions that made multiplicative ways of reasoning explicit. In particular, there was an opportunity to problematize the reports that noted the number of data values above a T-cell count of 525 or 550 (e.g. direct additive comparisons were insufficient as shown in Figure 7), creating a need for stronger forms of argument.

[Insert Figure 7 here]
As an example, just giving the count of data values above 525 (e.g., reasoning additively) would lead one to conclude that the traditional treatment protocol was more effective. It is only when that number is taken as a proportion or a percentage of the total that an adequate argument can be made (e.g., reasoning multiplicatively).

In making these a priori decisions about how to sequence the students’ reports, a conscious decision was made among the research team to start with qualitative comparisons and then move to quantitative ones. The purpose in this structuring was to start with analyses based on perceptual patterns in the data that provided a means for all students to engage in the critique. Then building from the students’ inscriptions, I hoped to initiate shifts toward more sophisticated inscriptions, thereby providing an opportunity for all students to participate in more acceptable forms of mathematical argumentation. This in turn could set the stage for argument as proof. This sequencing can be contrasted with introducing a four equal groups inscription and spending the class period explaining it to those who do not understand. The former alternative provided the means of supporting the students’ developing understandings.

I began the whole-class discussion the next day by asking the students first to decide if they could understand each report and then decide if they thought the inscription was an “adequate” way to represent the data in order for someone who had not seen the data sets to make a decision. In doing so, I was attempting to shift the conversations from simply providing explanations of methods, to critiques that raise the standard for argumentation. To facilitate this process, I took the students’ inscriptions and reproduced them on chart paper. The creation of these artifacts allowed the class to be able to see the reports clearly and removed ownership during the discussion. While the authoring
students’ might have been able to determine which inscription was theirs, it was also the case that other students in the class could have solved the task in the same way. For this reason, the students did not attend to whose solution was being critiqued.

The discussion began with Janine giving her assessment of the first report. She determined that the report was adequate since she could tell “where the range is starting and ending” and “where the majority of the numbers are.” This justification created a problem for David who did not understand what Janine meant by the term majority. In the ensuing discussion, I authorized the turn taking as the students attempted to answer David’s question.

**Janine:** Where most of the numbers were.

**David:** Where most of the numbers are…

**Kay:** Shanine can you help?

**Shanine:** Like, when she talks about like when she says, like when you say where the majority of the numbers were, where the, where the point is, like, you see where it goes up?

**Kay:** I do see where it goes up.

**Shanine:** Yeah, like right in there, that’s where the majority of it is.

**Kay:** Ok.

**David:** The highest range of the numbers?

**Shanine:** Yeah.

**Kay:** The highest range?

**Shanine:** Oh, no

**Val:** No
Kay: Val?

Val: However many people were tested, that’s where most of those people fitted in, in between that range.

From the comments, it appeared that for some of the students the inscription was a representation of the distribution of the data. In other words, they could not determine where the data were with respect to the inscription. In particular, Janine referred to “where most of those people fitted in.” This was important since the goal was that the students build initial understandings of distributions from perceptual patterns in the data.

After this discussion, I introduced the next two reports as shown in Figure 8.

[Insert Figure 8 here]

Both of the groups generating these reports had initially structured the data in a similar manner using the computer tool. One had partitioned the data at 525 and the other at 550. Both had then reasoned about the number of data values on either side of the cut point, albeit one in qualitative terms (e.g. majority) and the other in quantitative terms. At this point, Val asked why the one group had chosen to partition at 550. She noted that 550 did not represent the median so she could not determine its importance. Here Val was asking for a backing for the warrant. Again, using Toulmin’s scheme, we note that the students had analyzed the data to reach a conclusion. Their warrant consisted of looking at the number of data values above and below a T-cell count of 550. What was not obvious from the report was the choice of 550. In other words, the report had not included a backing for the warrant. Val wanted to understand why the group had decided to partition the data at a T-cell count of 550 since it did not represent any particular value
of the data set such as the median. The norm that had become instantiated in the classroom was that of justifying in terms of the context or the way the data was distributed. Val was making it clear that she could not ascertain the justification from the report. This is evidence that the norms for acceptable mathematical arguments had shifted again.

At this point, one of the students, Meg, who had solved the task in a similar manner noted that, when working on the computer tool, she noticed that the data “lined up straight on 550.” I recast this as basing the cut point on a natural break in the data. This was an important concept. One of the goals of the design experiment was that students would view distributions in terms of shape. Here Meg was offering an explanation based on the shape or breaks in the data as inscribed on the tool. For that reason I wanted to highlight both the solution and Meg’s explanation. This is a significant exchange because it points to a shift in the norms for argumentation. The students would not accept an argument whose backing could not be justified in terms of the analysis of the data.

It is important to note that Val asked for the backing for the choice of 550 and not 525. In looking at the inscription with a cut point at 525, it could be argued that the data were partitioned on the basis of the midpoint of the extremes (e.g. finding the average of the maximum and minimum value). In previous classes students had attempted to use the midpoint of the two extreme data values as a statistic for describing data sets. They called this value the “middle of the range.” I had worked to problematize this way of reasoning, as it has no statistical value in terms of distributions. In this episode, however, the students accepted it as a valid cut point since it coincided with the break in the hills.
Although aware of how the students had reasoned in creating the inscription, I appropriated the inscription in order to allow the opportunity to compare and contrast the qualitative and quantitative inscriptions shown in Figure 8.

It in fact proved to be a pivotal move to juxtapose the second and third reports. Although I had recast the two inscriptions as partitioning the data around the break in the hills, one group described the results in terms of where the majority of the data fell and the other used the computer tool to determine the number of data values that fell above and below the cut point. Juxtaposing these two reports made it possible to problematize the use of direct additive comparisons. This was facilitated by Missy noting that the second of these two reports would be more confusing, “since the old program has more numbers than the new program.” I realized the importance of this contribution and recast it so that all members of the class could focus on the possible problem.

Kay: Oh. So it looks like that there’s more. They had 56 that were above 525, and they only had 37?

Missy: So it’s like, I guess what I’m trying to say is it’s harder to compare them.

Kay: What about what Missy said? She just said there were more people in the old program so if you actually looked at the actual numbers of people, you find out that they had 56 that were in this upper range which is where we want to be and these only had 37. So somebody might say the old program was better because there were more. Janine?
As the discussion continued, students were able to acknowledge the problematic nature of the report, but could determine a solution. Finally, at Ken’s direction, I created a modified graph of the data as shown in Figure 9.

[Insert Figure 9 here]

At this point, students offered a variety of ways to reason about the numbers.

Brad: But then there’s more people with the old program than there is with the new program.

Kay: Juan.

Juan: Then you see that there’s 37 is more than half over 525 and 56 is not more than half of 130…more of them on the bottom than on the top.

Student: I don’t understand.

Kay: Juan?

Juan: Ok, you see how 37 is more than half of 9 and 37 together? But 56 is not more than half of 30, 130 and 56 put together. There’s more on the bottom one than on the top one.

Kay: Ok, who can help me out with that, who can say that a different way so that I might could understand that? Will, can you say it a different way?

Will: Well, in that situation it wouldn’t matter how many people were in there because see like,

Kay: Big voice, Will.

Will: On the bottom one you have, see what Juan was saying there’s more than there is below 525 and so that means that that one is better because the top one it doesn’t even have close to half of what the one below 525 is on that one. So that means
that if, if that was the same amount of people it had like, if they both had the same amount of people and, but, and they had the numbers and everything, and this one, the bottom one was a however much more that of…

Brad had an intuitive understanding of the problem but was unable to clarify other than to point to the difference in the number of patients in each program. Juan reasoned that since over half of the patients in the experimental treatment protocol had T-cell counts greater than 525 and less than half of the patients in the traditional treatment protocol had T-cell counts that high, the experimental was better. Although the nature of their arguments was not sophisticated, the students acknowledged the need to clarify the apparent problem in order for the inscription to be acceptable.

The last report to be shared in the whole-class discussion involved students structuring the data into four equal groups as shown in Figure 10.

[Insert Figure 10 here]

After I posted the report on the white board, Brad stated that he felt it was adequate because the extreme data values were similar in both data sets and “with the 4 equal groups, you can tell where the differences in the 4 groups.” In his explanation, Brad pointed to two mathematically significant issues, both of which were of interest to me. The first concerned how to interpret the four equal groups graph. Brad noted that the new treatment was better because “the three lines for the equal groups” from the experimental were above 525 compared to “only one of them” for the traditional treatment. Although somewhat difficult to understand, I interpreted Brad’s explanation to mean that he noted that 75% of the experimental data (i.e. three lines) was above 525
whereas only 25% of the traditional data (i.e. one line) was above 525. It was my belief that many of the students in the class did not understand Brad’s offered explanation. Therefore, teasing out the meaning of the graph was an important aspect of the continuing discussion.

The second mathematically significant issue concerned Brad’s reference to the fact that since the extreme data values (maximum and minimum) were near the same on the two data sets, the comparison was easier. This was echoed by Mark who noted that in the past they had not been similar (i.e. “like kind of crooked” and “this time it’s easier to see because it’s right under each other.”) The issue of similar extreme data values had occurred on prior tasks and in some cases the students had stacked the two inscriptions and aligned the minimum data values in both data sets, even when they were not the same. This was clearly problematic from my perspective as the values were on a scale that could not simply be adjusted so that these two values were always the same. Here the students noted that this inscription was easier to read because of the similarity in extreme data values of the two data sets, in particular the minimum value. Although it would have been difficult for me to pose a comparison problem in this class session where the minimum data values were very different in order to make that an explicit topic of conversation, I nonetheless noted it as an issue that needed to be addressed.

Clarifying the understanding of the four equal groups graph was therefore the primary consideration. As I considered how to continue, Mark suggested placing numbers in the intervals to indicate the number of data points in each. I found this very significant, as the purpose of the four equal groups graph was to remove the numbers and thus the problems of comparing data sets with unequal N’s. However, it was obvious
from the comment that the intervals did not represent proportions of the data for him. I therefore judged that at this point other students were unable to read the four equal groups inscription. Not only did it not represent the data, they had no resources for recreating the data.

As the lesson continued, I asked the students to calculate the actual number of data points that would fall within each interval in each data set. As they did, I wrote each of these values within the intervals of the four equal groups inscription as shown in Figure 11.

[Insert Figure 11 here]

Once these numbers were visible, the students commented that they did not help because there was not the same number of people in each treatment. This was similar to the argument that Missy made about the third inscription. I then asked the students if they knew what percentage of the data was in each interval. In their answer the students were able to clarify that they understood that 25% of the data set fell within each of the four intervals. It could be argued that this was facilitated by the fact that they had just used calculators to find 25% of each data set in order to place the numbers in the intervals. Nonetheless, I then noted on the inscription, at the students’ direction, that 75% of the patients in the new treatment program were in the same range of T-cell counts as only 25% of the patients in the old treatment (see Figure 12).

[Insert Figure 12 here]

For many of the students, the process of calculating the numbers and then going back to the percentages was necessary to build an understanding to underlie the inscription. They needed to know where the data were behind the inscription. The four equal groups
inscription was not interpreted as providing information on the distribution of the data in the same way the first inscription was. It was therefore important for them to understand how the data were being represented.

Perhaps the most critical aspect of this particular episode was the a priori selection and sequencing of the students’ inscriptions of the data to be discussed. During this whole-class discussion, a planful attempt was made to sequence the solutions in order to provide a progression toward more sophisticated ways of structuring the data while simultaneously eliciting more sophisticated forms of mathematical argument. A review of the sequence of solutions finds the whole-class discussion initially focused on reports that structured the data by partitioning around the hills. The first two solutions offered a qualitative distinction between the two data sets by focusing on the location of the majority of the people. The third, however, gave a quantitative distinction that provided the opportunity for a problem to arise from direct additive comparisons. This, in turn, offered me the opportunity to problematize the direct additive comparisons and shift the discussion to more sophisticated ways to reason about the data sets. The four-equal-groups solution was then a logical next step in the progression. In this way, the inscriptions served as tools for reorganizing the students’ activity, providing me with the means of supporting their development of more sophisticated mathematical arguments (Dorfler, 1993; Kaput, 1994, Meira, 1998; Pea, 1993).

Throughout the twelve weeks of the teaching experiment, there were continual shifts in the norm of what counts as an acceptable mathematical argument. Although there were necessarily cycles of learning during which new inscriptions (e.g. the four equal groups) emerged and required more detailed forms of explanation of versus
argumentation with, nonetheless, reasoning with the inscriptions became normative and created the discourse space for arguments to take the stance of proof.

Discussion

The importance of deliberately facilitated whole-class discussions is often subjugated to a Romantic Constructivist impression of classrooms in which teachers “step back” while students construct their own understandings. My view of the role of the teacher in this era of reform places a high premium on the importance of managing mathematical discussions (cf. McClain, 2004). Significant mathematical discussions provide resources for supporting students’ emerging understandings while taking their initial understandings as starting points. A dilemma in this approach centers on a concern for offering means of support while not providing so much support that the students’ activity becomes that of following anticipated cues. For this reason, teaching becomes a more complex endeavor while appearing to be less intrusive.

In the episodes in this paper, my learning is evident. Although my decision-making process was informed by my anticipations of the students’ ways of reasoning on each task, initially I had trouble distancing myself from my own ways of reasoning and taking account of the students’ current understandings. For that reason, I often imposed more sophisticated meanings on their activity, thereby creating problems in my interpretations of their analyses. I had the advantage (or disadvantage) of seeing the endpoint of the trajectory and being able to envision how I might traverse the route. What I did not account for were the diverse ways of reasoning that the students employed in
their route. As I began to capitalize on their ways of reasoning, my participation supported (instead of hindered) their mathematical learning.

Concurrently, the students were able to engage in progressively more sophisticated analyses. However, it is questionable whether or not their arguments ever constituted proof. Their justifications were grounded in the question at hand and over the course of the design experiment, they developed, in Toulmin’s terms, the need and understanding for mathematically based backings for their warrants. For the students, their discussions and critiques of the varied solutions took the form of refinement, not of proof.

In Simon’s (1997) description of the Mathematics Teaching Cycle he notes the “inherent tension between responding to the students' mathematics and creating purposeful pedagogy based on the teacher's goals for student learning” (p. 76). Purposeful pedagogy in these episodes would entail an explicit focus on mathematical argumentation. This cannot be subjugated to completion of process or finding a solution. It must be an intricate part of the mathematical process. One could argue that statistical data analysis is the perfect domain in which to concurrently develop mathematical concepts and appropriate forms of argumentation and proof. I would agree and point to the importance of students’ ways of reasoning in supporting the emergence of productive argumentation while acknowledging that the mathematics must be the central focus in all decisions.

Notes
1. The research team members involved in the teaching experiment included myself, Paul Cobb, Koeno Gravemeijer, Maggie McGatha, Lynn Hodge, Jose Cortina, Carla Richards, and Cliff Konold.

2. Although I assumed primary responsibility for teaching, daily debriefing sessions were held with the research team permitting me extended expertise. I do, however, take full responsibility for the miscommunications during class and credit my colleagues, especially Paul, for my professional growth over the course of the design experiment.

3. As part of my role as teacher in the classroom teaching experiment, I kept a daily journal that contained my observations and reflections from the classroom. It was here that I made notes of my understandings of the students’ ways of reasoning over the course of the twelve weeks of the teaching experiment. This journal then became a data source in the course of analysis.

4. The features on the computer tool were designed to allow students to capture a subset of the data values in order to make decisions about the range or variability. The value bar feature was intended to offer students a means of partitioning the data, identifying the median, or estimating the mean. Our intent was that the students would use these features of the tool as they reasoned about how to structure the data in order to make a judgment or comparison. In this way, the tool offered a means of supporting the mathematical agenda.

5. Here I want to make a distinction between argument as disagreement, argument as refinement, and argument as proof. The changes in task allowed the argument to shift from disagreement to refinement.
6. This decision was made a priori in consultation with members of the research team.
References


Figure 1. Data displayed in the first computer tool.
Figure 2. Battery data with range feature shown capturing the top ten values.
Figure 3. Battery data with value bar and range feature shown.
Figure 4. Toulmin’s scheme for argumentation.
Figure 5. Data inscribed in the second computer tool.
Figure 6. An inscription resulting from partitioning the data.
Figure 7. AIDS data partitioned at 525.
Report #2:

The new drug was better than the old. The majority of the old ones are behind 550 and the majority of the new drug was in front of 550.

---

Report #3:

<table>
<thead>
<tr>
<th>Experimental treatment</th>
<th>Traditional treatment</th>
</tr>
</thead>
<tbody>
<tr>
<td>225 - 525</td>
<td>225 - 525</td>
</tr>
<tr>
<td>9</td>
<td>130</td>
</tr>
<tr>
<td>525 - 850</td>
<td>525 - 850</td>
</tr>
<tr>
<td>37</td>
<td>56</td>
</tr>
</tbody>
</table>

Figure 8. AIDS data as noted in reports two and three.
Figure 9. Graph created from partitioned AIDS data.
Figure 10. Inscription of the AIDS Protocol Data Organized into Four Equal Groups.
When is an Argument just an Argument?

Experimental Treatment

<table>
<thead>
<tr>
<th></th>
<th>12</th>
<th></th>
<th>12</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td></td>
<td>300</td>
<td></td>
<td>400</td>
</tr>
<tr>
<td>500</td>
<td></td>
<td>600</td>
<td></td>
<td>700</td>
</tr>
<tr>
<td>800</td>
<td></td>
<td>900</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td></td>
<td>1100</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Traditional Treatment

<table>
<thead>
<tr>
<th></th>
<th>46</th>
<th></th>
<th>46</th>
<th>46</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td></td>
<td>300</td>
<td></td>
<td>400</td>
</tr>
<tr>
<td>500</td>
<td></td>
<td>600</td>
<td></td>
<td>700</td>
</tr>
<tr>
<td>800</td>
<td></td>
<td>900</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td></td>
<td>1100</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 11. Four Equal Groups Inscription with Number of Data Values Noted.
When is an Argument just an Argument?

Figure 12. Four Equal Groups Inscription with Percentages Marked.